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CONTENTS

A theoretical investigation is given of the phenomena arising when vertically polarized radio waves are propagated across a boundary between two homogeneous sections of the earth's surface which have different complex permittivities. The problem is treated in a two-dimensional form, but the results, when suitably interpreted, are valid for a dipole source. The earth's surface is assumed to be flat.

In the first part of the paper one section of the earth is taken to have infinite conductivity and is represented by an infinitely thin, perfectly conducting half-plane lying in the surface of an otherwise homogeneous earth. The resulting boundary-value problem is initially solved for a plane wave incident at an arbitrary angle; the scattered field due to surface currents induced in the perfectly conducting sheet is expressed as an angular spectrum of plane waves, and this formulation leads to dual integral equations which are treated rigorously by the methods of contour integration. The solution for a line-source is then derived by integration of the plane-wave solutions over an appropriate range of angles of incidence, and is reduced to a form in which the new feature is an integral of the type

$$G(a, b) = b\,e^{ia^2}\int_a^\infty \frac{e^{-i\lambda^2}}{\lambda^2+b^2}\,d\lambda,$$

where a and b are in general complex within a certain range of argument.

The case when both the transmitter and receiver are at ground-level is considered in some detail. If the receiver is a large 'numerical distance' from the transmitter, further simplification is possible; the results then agree with some previously given by Feinberg, whose method, however, was quite different. The practical adequacy of Millington's graphical technique for deriving attenuation curves of the ground-to-ground field is demonstrated, and the possibility of an increase of field-strength with distance is confirmed. This 'recovery effect' is illustrated by a numerical example in which the phase curve is also shown to rise steeply just beyond the boundary, indicating a phase velocity in this region much greater than that in free space.

A different approximate form of the general solution is obtained when the transmitter and receiver are sufficiently elevated; this is used to indicate the validity of the application of height-gain factors over an appreciable range of heights.

In the second part of the paper the restriction that one of the earth media should be perfectly conducting is waived. A condition, usually met in practice, is assumed, namely, that the modulus of the complex permittivity of each section of the earth is large. Approximate boundary conditions are then likely to be valid, and their introduction makes possible an analytical treatment on the same lines as before. The solution is again reduced to a form only involving, apart from standard features, integrals of the type $G(a, b)$. Various features of the expression for the ground-to-ground field are examined; in a numerical example the attenuation and phase curves are given, the former being compared with the results of an experiment previously reported by Millington and the agreement shown to be good. The different approximate form of the solution when the transmitter and receiver are sufficiently elevated is briefly considered.

Finally, some ramifications of the theory are outlined.

1. Introduction

1·1. *The genesis and nature of the problem*

The theory of the propagation of radio waves over a smooth, finitely conducting, *homogeneous* earth, neglecting atmospheric and ionospheric effects, is now well matured. The first correct discussion of the case when the distances from the transmitter are sufficiently small for the earth's surface to be considered flat was given by Sommerfeld (1909) over forty years ago, and independent fundamental treatments adopting the model of a spherical earth, appropriate for greater distances, have been presented more recently by Vvedensky (1935, 1936, 1937), Van der Pol & Bremmer (1937, 1938, 1939) and Eckersley & Millington (1938). In practice, however, the earth's crust may be significantly *inhomogeneous*, and the need has long been felt for a theory which would at least take into account the more pronounced variations in the electrical properties of the terrain over the region of interest. The most striking features of the complicated general problem thus presented appear when vertically polarized ground-waves are transmitted across a boundary of discontinuity, such as a coast-line, which separates two media of markedly different characteristics. A theoretical treatment of this aspect is given in the present paper.

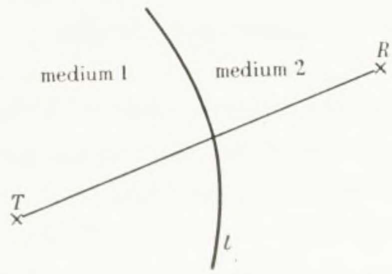

FIGURE 1. Propagation path across a boundary separating two different media.

Figure 1 is a plan of a smooth area of the earth's surface in which the boundary line l separates the (homogeneous) media 1 and 2, on which are situated, respectively, the transmitter T and receiver R. There are three branches of radio technique in which it may be necessary to consider the consequences of a physical model of this type:

(*a*) *Field-strength assessment.* The service area of a transmitter depends profoundly on the nature of the ground, and the presence of marked inhomogeneities in the earth's surface is therefore of practical importance. The first suggestion for estimating the variation of field-strength with distance along a composite path was made by P. P. Eckersley (1930), and latterly considerable attention has been given to this question, which is sometimes referred to as that of 'mixed-path attenuation'.

(*b*) *Direction finding.* In certain circumstances it has been found that the apparent bearing of T from R measured by standard radio methods can be appreciably different from the true bearing. This phenomenon was first noticed by T. L. Eckersley (1920), and is commonly known as 'coastal refraction'.

(*c*) *Navigation.* The operative principle of some modern radio navigation equipment is the interpretation of accurate phase measurements. The significance, in this connexion, of the variation of the phase velocity of waves propagated over a homogeneous earth was stressed by Norton (1947) and Ratcliffe (1947*a*), and the corresponding effect with a composite path, which is complicated by the distortion of the phase fronts arising from the discontinuity at l, must also be considered.

These issues are, of course, interlinked, and a complete solution of the boundary-value problem illustrated in figure 1 would apply to all three. The analytical difficulties, however, are formidable, and in this paper the mathematical discussion is confined explicitly to the two-dimensional case in which the boundary between the media is straight and the transmitter is an infinitely long (vertically polarized) line-source parallel to it. On the other hand, it seems very probable that, suitably interpreted, the solution may be applied to the problem when the source is a more practical aerial such as a vertical dipole. This contention (known to be true for a *homogeneous* earth: see §3 and also Booker & Clemmow (1950*b*)) is evidently most reasonable at 'normal incidence', that is, when TR is perpendicular to l; and in consequence the treatment here is chiefly directed at (a), where the main features are expected to be independent of the angle of incidence, rather than at (b) or (c), which would require a more specific consideration of oblique incidence. Nevertheless, the variation of phase as well as amplitude is established in the solution to be given, and it should therefore act as some guide in these latter problems; particularly is this so since subsequent work indicates, as also does that of Feinberg (1946), that to a marked extent the field along each radial line from the transmitter depends only on distance measured along that line and not on its direction relative to the boundary.

In what follows it is assumed that the earth's surface is flat. As in the theory of a homogeneous earth, the analysis is governed by this assumption, and cannot therefore be extended to deal with the case of a spherical earth, for which a quite distinct treatment would be required.

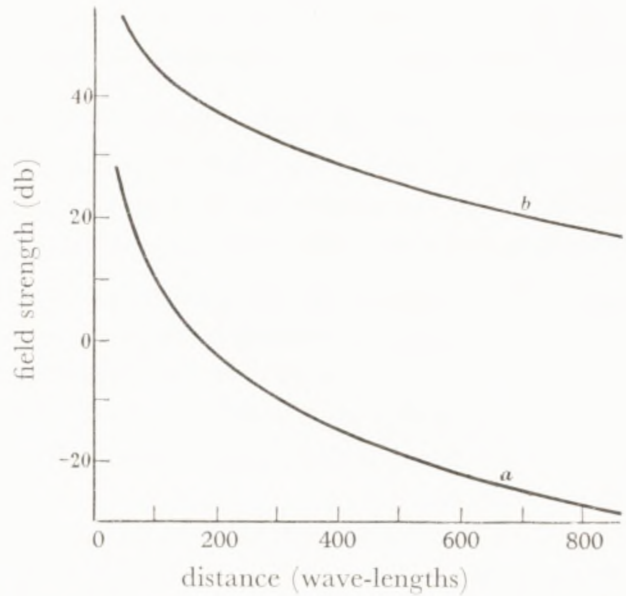

FIGURE 2. Field-strength (in decibels above an arbitrary level) against distance (in wave-lengths) from the transmitter for homogeneous earths: (a) medium 1; (b) medium 2.

For practical purposes, the theory of propagation over the earth is conveniently expressed by means of graphs which show the variation of field-strength and phase with distance from the transmitter. The field-strength in decibels above an arbitrary level and the phase in degrees relative to that of the undisturbed free-space field of the transmitter are plotted against d/λ, where d is the distance and λ the wave-length. It should be noted that all such curves in this paper are referred to a transmitter for which the free-space field in the 'radiation region' falls off inversely as d; for a line-source this field is proportional to $1/\sqrt{d}$, and a further factor $1/\sqrt{d}$ must be introduced in adapting the two-dimensional analysis to a point-source. Two typical flat-earth attenuation curves are shown in figure 2; curve (a) is for a homogeneous earth of medium 1 and curve (b) for homogeneous earth of medium 2, say, where the modulus of the complex permittivity of medium 2 is much greater than that of medium 1. The present problem, in short, is to calculate the corresponding curve for an inhomogeneous earth when the ground between the transmitter and receiver consists of medium 1 in one section and medium 2 in the other. And similarly with phase. A question of particular interest is the possibility of an *increase* of field-strength with distance in the region just beyond the boundary, the 'recovery effect'.

1·2. *Previous work*

Of the various theoretical approaches to 'coastal refraction' and 'mixed-path attenuation', two pieces of work, quite distinct in character from each other, seem of major importance. One is an analytical approach, nominally directed at the former problem, initiated by Grünberg (1942, 1943) and developed by Feinberg (1944, 1945, 1946); the other, due to Millington (1949*b*), an 'engineering' method for the latter problem.

Grünberg showed that the adoption of approximate boundary conditions and a standard application of Green's theorem yield an integral equation for the normal component of **E** at the earth's surface. He considered the case of two earth media, one of which has infinite conductivity, separated by a straight boundary, and took the incident field to be a plane wave. While appreciating that his integral equation could be solved by the exact method of Wiener & Hopf (Titchmarsh 1937), he preferred an approximate treatment from which he established that the direction of propagation at a great distance beyond the boundary is the same as that of the incident wave. Grünberg's work was generalized by Feinberg in a series of papers of which the fourth (Feinberg 1946) treats this problem, but with the difference that a transmitter located at a finite distance from the boundary is introduced. The analysis is so manipulated that an assumed value may reasonably be substituted for the unknown field component under the integral sign; in this way the problem becomes one of integration, and limiting expressions are derived appropriate to various positions of transmitter and receiver. These latter important results have apparently attracted little attention in this country, and the present work was completed before they became known to the author.* As will be clear from §2·1, the method of this paper is quite distinct and the treatment in some respects complementary; on the other hand, such formulae as do correspond show complete agreement.

An entirely different approach has led Millington (1949*b*) to suggest a simple technique for deriving mixed-path attenuation curves, when the transmitter and receiver are both at ground-level, from the appropriate individual curves for homogeneous earths. His procedure has affinities with those of P. P. Eckersley (1930) and Somerville (Kirke 1949); but it is much more skilfully contrived than either of these, being designed, among other things, to satisfy the reciprocity requirement regarding the interchangeability of transmitter and receiver which these other two methods clearly violate; to this end it takes into account the one special result which can be deduced immediately from homogeneous earth analysis, namely the 'geometric mean formula', first given by T. L. Eckersley (1948, p. 78) specifically for the case of a spherical earth where the boundary is in the diffraction region of the transmitter and receiver, and shown by Millington to be more generally applicable. Millington's technique is based on arguments of a conjectural nature, but its predictions, including the possibility of a 'recovery effect', have proved to be in remarkably good agreement with experiments over a wide range of frequencies as described by Millington (1949*a*, *c*), Elson

* I am indebted to Mr J. J. Myers for drawing my attention to the paper of Feinberg's which is of particular relevance.

(1949), Millington & Isted (1950) and Bramslev (1949; see also the Discussion following Millington & Isted (1950)). It is therefore of practical importance to note that the present analysis indicates that field-strengths estimated by Millington's method are not likely to be appreciably in error, and an 'engineering' solution is thus given good theoretical backing, notwithstanding that its success appears to be to some extent fortuitous.

Finally, some tentative suggestions regarding a mechanism for coastal refraction, recently offered by T. L. Eckersley (1948, p. 97) and Ratcliffe (1947 *b*), should be mentioned. When first discussing this phenomenon Eckersley (1920) reasoned by analogy with ordinary refraction theory, but based his argument on the invalid concept of propagation due to Zenneck (1907); the later approach is similar in character, but invokes the correct analysis for a flat or curved homogeneous earth. Whatever value such ideas may prove to have will certainly be enhanced by considering them in terms of the present mixed-path solution, since this provides a much fuller description of the variation of phase across a coastline than has hitherto been available.

PART I. WHEN ONE MEDIUM HAS INFINITE CONDUCTIVITY

2. Generalities

2·1. *The idealized problem and method of solution*

As already stated, the mathematical attack is on the two-dimensional form of the problem in which we have a vertically polarized line-source parallel to a straight boundary. This model may be compared with the idealization suggested by Millington (1949 *b*) of axial symmetry about a vertical dipole.

In this first part of the paper we also specialize by the assumption that one of the media (medium 2) has infinite conductivity, and this medium is replaced by an infinitely thin, perfectly conducting, semi-infinite sheet situated in the interface of the air (regarded as free-space) and medium 1, the latter being taken to fill the complete region below the interface (figure 3). The assumption of perfect conductivity for an earth constituent may sometimes be justified, sea water, for example, often fulfilling this condition to an adequate degree of accuracy. Furthermore, under most practical conditions the radiation penetrates negligibly into the ground, so that the results given by the model of figure 3 are not likely to be significantly different from those obtained (were it possible) from a more realistic model in which medium 2 has a finite depth.

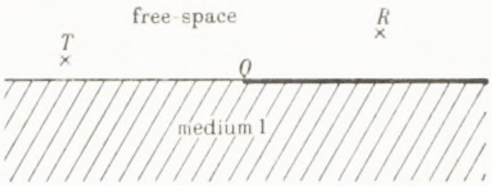

FIGURE 3. The model of the idealized problem.

Since the more general problem involving two arbitrary media is discussed in the second part of the paper, it is perhaps as well to state why it seemed desirable to begin with a particular case. In the first place, with the model of figure 3 an exact solution is possible; this is not so when both media are arbitrary, and it is then necessary to assume approximate boundary conditions at the outset of the analysis; such a procedure (adopted by Grünberg

& Feinberg) is, perhaps, open to objection, and it is reassuring to find that, when applied to the special case, it gives virtually the same result as the exact solution. Secondly, the geometric mean formula already mentioned refers, as is shown later, to circumstances in which 'ray theory' may be used with effective Fresnel reflexion coefficients of -1 for both media; a model in which the reflexion coefficient of medium 2 is always $+1$ is therefore of particular interest in that it represents a situation where these conditions are completely violated. Thirdly, the analysis is somewhat complicated and may be more easily followed by starting with the special case which furnishes some relatively compact formulae and a straightforward physical interpretation.

The problem illustrated in figure 3 has so far been regarded as a generalization of that of propagation over a homogeneous earth. It may also be thought of as a generalization of the famous problem, likewise first solved by Sommerfeld (1896), of diffraction by a perfectly conducting half-plane, to which it would revert if medium 1 were free-space; and from this point of view the recovery effect appears perhaps less remarkable than might otherwise be supposed. In the present case, however, the features commonly associated with diffraction are obscured by the fact that both the line-source and point of observation are very near the

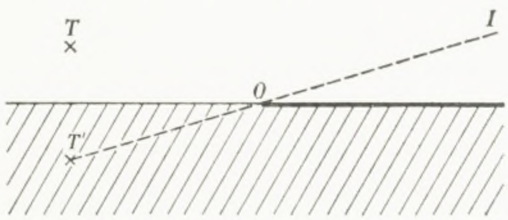

FIGURE 4. The line (OI) across which the field of 'geometrical optics' is discontinuous.

earth's surface, and this rules out the possibility of using any simple approximation of the Huygens-Kirchhoff type. On the other hand, as described below, an exact method of solution is available in which it is convenient to preserve the concepts of a 'geometrical optics' field and a 'diffraction' field. In figure 4, T' is the image of T in the earth's surface, and I is the point at infinity on $T'O$ produced; by definition, the geometrical optics field in the free-space region to the left of OI is that which would obtain for a homogeneous earth of medium 1; in the region to the right of OI, that which would obtain for a homogeneous perfectly conducting earth; the residue of the total field is the diffraction field, which, in particular, has a discontinuity across OI counterbalancing that of the geometrical optics field. A diffraction field can often be interpreted as arising from a fictitious source located at the diffracting edge; in the present case it may be thought of as some disturbance due to the boundary, although it cannot be conceived in terms of a line-source at O for positions of the receiver as close to OI as those with which we are concerned; for these positions the diffraction term is comparable with that of geometrical optics, and its evaluation forms the chief part of the analysis.

It is evident that the mathematics of our problem must represent a fusion of (exact) diffraction theory and propagation theory. The fundamental contribution in each of these fields was made by Sommerfeld, but the methods originally used bear no relation to one another and cannot be readily generalized in the way which we require. On the other hand, a powerful technique which expresses any electromagnetic field as an angular spectrum of plane waves (Booker & Clemmow 1950*a*) has been shown to be effective both in the theory of propagation over a flat, homogeneous earth (Booker & Clemmow 1950*b*), and in rigorous diffraction theory (Clemmow 1951). The method is appropriate to the present problem and is applied in this paper.

In § 3 some results from the theory of propagation over a homogeneous, flat earth are briefly derived in a way specifically suited to the subsequent discussion; attention is drawn to the explanation of the sign error in Sommerfeld's 1909 paper which has given rise to a controversy recently revived by Epstein (1947) and others. It is then shown (§ 4) that for a composite path the geometric mean formula can only be justified on a ray-theory basis together with the assumption that both media have effective reflexion coefficients of -1. In § 5 the problem of a plane wave incident on the interface shown in figure 3 is expressed in terms of dual integral equations and the formal solution obtained. The corresponding solution for a line-source is deduced by representing a cylindrical wave as an angular spectrum of plane waves, and some reduction is carried out (§§ 6, 7). The special configuration in which both the transmitter and receiver are on the earth's surface is considered more closely, and agreement found with Feinberg's results in limiting cases; the recovery effect is illustrated by a numerical example in which the field-strength and phase curves are plotted (§ 8). In § 9 the different approximate form which the solution may assume when the transmitter and receiver are sufficiently elevated is examined with particular reference to the use of height-gain functions. The reason for the success of Millington's technique when applied to the present problem is analyzed in § 10.

2·2. *Some remarks on notation*

The following remarks are intended as a general guide, and symbols not listed below are defined as they arise in the text.

With Cartesian co-ordinates x, y, z, the earth's surface is taken as the plane $y = 0$; the origin is located at O, as in figure 3, the z-axis being along the boundary and the two-dimensional field independent of z. Polar co-ordinates r, θ are also used, where $x = r\cos\theta$, $y = r\sin\theta$, $0 \leqslant \theta \leqslant 2\pi$. Other co-ordinates are:

$x_0, y_0;\ r_0, \theta_0$	co-ordinates of the transmitter T
R	distance from T
S, ψ	polar co-ordinates measured from the image T'
$d = \lvert x - x_0 \rvert = \lvert S\cos\psi \rvert$	horizontal distance from T
$R_1 = r + r_0$	

The configuration is illustrated in figure 5.

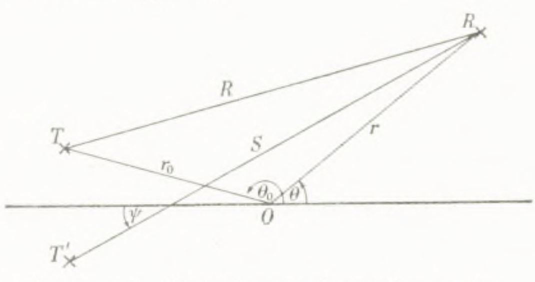

FIGURE 5. Notation and configuration.

Rationalized m.k.s. units are used, and a time factor $\exp(\mathrm{i}\omega t)$ suppressed throughout. We write

ϵ_1	permittivity in farads/metre,
μ	permeability in henrys/metre,
σ	conductivity in mhos/metre,
$\epsilon = \epsilon_1 - \mathrm{i}\sigma/\omega$	complex permittivity,
$k = \omega\sqrt{(\epsilon\mu)}$	propagation constant,
$Z = 1/Y = \sqrt{(\mu/\epsilon)}$	intrinsic impedance,

these symbols referring to free-space (for which $\sigma = 0$, $\epsilon = \epsilon_1$), and the same symbols with dashes denoting the corresponding quantities for the earth (medium 1).

A two-dimensional electromagnetic problem is essentially scalar, the vertically polarized field $\mathbf{H} = (0, 0, H_z)$, $\mathbf{E} = (E_x, E_y, 0)$ being expressible in terms of H_z via Maxwell's equations

$$E_x = \frac{Z}{\mathrm{i}k}\frac{\partial H_z}{\partial y}, \quad E_y = -\frac{Z}{\mathrm{i}k}\frac{\partial H_z}{\partial x}. \tag{1}$$

Formulae are therefore given for H_z only. In propagation theory it is perhaps more usual to work in terms of the component of $\mathbf{E}$ normal to the earth's surface, but for the radiation field at sufficiently small angles of elevation

$$E_y \doteqdot ZH_z, \tag{2}$$

and so the distinction is unimportant. For convenience we suppose that the transmitter (line-source) has a circular polar diagram, though an arbitrary polar diagram could equally well be considered; its undisturbed field in free-space is then given by

$$H_z = \sqrt{(\tfrac{1}{2}\pi)}\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\,H_0^{(2)}(kR) \sim \frac{\mathrm{e}^{-\mathrm{i}kR}}{\sqrt{(kR)}}. \tag{3}$$

Superscripts attached to the field components have the following significance

i, r, s	*incident*, *reflected*, *scattered* field respectively;
g, d	*geometrical optics*, *diffraction* field respectively;
p	field associated with an incident *plane* wave.

Three abbreviations, although defined in the text, are listed here for reference; they occur frequently in the analysis:

$$F(a) = \mathrm{e}^{\mathrm{i}a^2}\int_a^\infty \mathrm{e}^{-\mathrm{i}\lambda^2}\,\mathrm{d}\lambda,$$

$$K(a) = 1 - 2\mathrm{i}aF(a),$$

$$G(a,b) = b\,\mathrm{e}^{\mathrm{i}a^2}\int_a^\infty \frac{\mathrm{e}^{-\mathrm{i}\lambda^2}}{\lambda^2 + b^2}\,\mathrm{d}\lambda.$$

Finally, we note that $S(\phi)$ is used for the 'steepest descents' path of integration passing through the real angle ϕ; no confusion should arise between this and the S defined above.

3. Some results from the analysis for a homogeneous earth

In this section we briefly treat the problem of propagation over a *homogeneous* earth. The purpose is to obtain the standard results by a method and in a form with which direct comparison can be made when we come to the mixed-path problem.

We consider a line-source, specified by (3), situated in free-space above a homogeneous earth of finite complex permittivity occupying the region $y<0$, and we are interested in the radiation field at small angles of elevation ($kR \gg 1, \psi$ small).

The basic formula for the factor by which the free-space field must be multiplied to obtain the field in the presence of the earth is (22). When the transmitter and receiver are both at ground-level it reduces to (24). At large 'numerical distances' useful simplified results are (25) for this latter case, and the ray-theory formula (26), applicable when the transmitter and/or receiver are sufficiently elevated. Also of great value are the 'height-gain' factors implicit in (31).

3·1. *The general solution*

Confining the discussion to the region $y \geq 0$, we consider a plane wave

$$H_z^{pi} = e^{ikr\cos(\theta-\alpha)}, \tag{4}$$

which is incident on the earth's surface at an angle α. This gives rise to the reflected wave

$$H_z^{pr} = \rho(\sin\alpha)\, e^{ikr\cos(\theta+\alpha)}, \tag{5}$$

where $$\rho(\sin\alpha) = \left\{\sin\alpha - \frac{1}{n}\sqrt{\left(1-\frac{\cos^2\alpha}{n^2}\right)}\right\} \Big/ \left\{\sin\alpha + \frac{1}{n}\sqrt{\left(1-\frac{\cos^2\alpha}{n^2}\right)}\right\} \tag{6}$$

is the earth's Fresnel reflexion coefficient and

$$n = k'/k = \sqrt{\{\epsilon_1'/\epsilon - i\sigma'/(\omega\epsilon)\}}, \tag{7}$$

assuming that the permeability of the earth is the same as that of free-space. In order to derive the field for a line-source situated at r_0, θ_0, we express the incident cylindrical wave (3) as an angular spectrum of plane waves of the type (4). Introducing the appropriate phase factor $\exp\{-ikr_0\cos(\theta_0-\alpha)\}$, we have

$$H_z^i = \frac{e^{-\frac{1}{4}i\pi}}{\sqrt{(2\pi)}}\int_C e^{-ikr_0\cos(\theta_0-\alpha)}\, e^{ikr\cos(\theta-\alpha)}\, d\alpha, \quad \text{for } y \leq y_0, \tag{8}$$

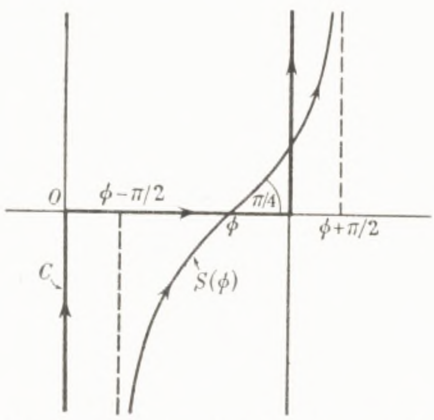

Figure 6. Paths of integration in the complex α-plane.

where the path C is shown in figure 6. C is only one of an infinite set of equivalent paths, but is particularly convenient in that along it $\cos\alpha$ ranges over real values from $+\infty$ to $-\infty$ (Booker & Clemmow 1950*a*, *b*; Clemmow 1951). The reflected wave corresponding to (8) is evidently found by multiplying (5) by $(2\pi)^{-\frac{1}{2}}\exp\{-\mathrm{i}[kr_0\cos(\theta_0-\alpha)+\frac{1}{4}\pi]\}$ and integrating with respect to α along C; thus

$$H_z^r = \frac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{\sqrt{(2\pi)}}\int_C \rho(\sin\alpha)\,\mathrm{e}^{\mathrm{i}kS\cos(\psi+\alpha)}\,\mathrm{d}\alpha. \tag{9}$$

If we write

$$\rho'(\sin\alpha) = \rho(\sin\alpha)-1 = -\frac{2}{n}\sqrt{\left(1-\frac{\cos^2\alpha}{n^2}\right)}\Big/\left\{\sin\alpha+\frac{1}{n}\sqrt{\left(1-\frac{\cos^2\alpha}{n^2}\right)}\right\}, \tag{10}$$

the complete field becomes

$$H_z = \sqrt{(\tfrac{1}{2}\pi)}\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\{H_0^{(2)}(kR)+H_0^{(2)}(kS)\}+\Delta(S,\psi), \tag{11}$$

where

$$\Delta(S,\psi) = \frac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{\sqrt{(2\pi)}}\int_C \rho'(\sin\alpha)\,\mathrm{e}^{\mathrm{i}kS\cos(\psi+\alpha)}\,\mathrm{d}\alpha. \tag{12}$$

The term $\Delta(S,\psi)$ in (11) is the field which must be added to that pertaining to a perfectly conducting earth, and its evaluation constitutes the core of the problem. It has been shown elsewhere (Booker & Clemmow 1950*b*) that this term is essentially equivalent to the free-space field of a Zenneck wave diffracted under the image line T' (figure 5). A Zenneck (1907) wave is a plane wave incident on the earth at the Brewster angle α_B, defined by

$$\tan\alpha_B = 1/n, \tag{13}$$

and an appeal to the well-known formula of edge-diffraction theory leads to the required result for the radiation field. For our present purposes, however, we proceed to an approximate evaluation of (12) by using an extension of the standard method of integration by steepest descents. The technique, suggested by Pauli (1938), was applied to the three-dimensional form of the present problem by Ott (1943); it has been considered in some detail by the author (Clemmow 1950*b*) and proves indispensable in the sequel.

The first step is to displace the path of integration C in (12) to that of steepest descents. We denote by $S(\phi)$, where ϕ is any real angle between 0 and π, the path, shown in figure 6, over which the new variable of integration

$$\tau = \sqrt{2}\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\sin\tfrac{1}{2}(\alpha-\phi) \tag{14}$$

traverses real values from $-\infty$ to $+\infty$. Then the required path of steepest descents is $S(\pi-\psi)$, the 'predominant' value of α (the saddle-point) being clearly $\pi-\psi$, as would be expected from physical considerations. Now the singularities of $\rho'(\sin\alpha)$ are branch points at

$$\cos\alpha = \pm n, \tag{15}$$

and poles at

$$\sin\alpha = -\sin\alpha_B = -\frac{1}{\sqrt{(1+n^2)}}; \tag{16}$$

and since $0 \geqslant \arg n \geqslant -\frac{1}{4}\pi$, these are located somewhat as in figure 7. It follows that, in displacing C to $S(\pi-\psi)$ (which cuts the real axis at an angle of 45°) no poles are captured. On the other hand, a branch-point may be crossed, and certainly is in the case of interest when ψ is small (or nearly equal to π; unless otherwise stated we assume without loss of generality that $\psi \leqslant \frac{1}{2}\pi$). Strictly, therefore, an integral round the corresponding branch-cut should be included; however, we follow the standard practice and neglect this contribution, a procedure which is justified either by the fact that n has an appreciable imaginary part, or, when this is not the case, by the fact that $|n| \gg 1$ (Ott 1942). Equation (12) may therefore be written

$$\Delta(S, \psi) = \frac{e^{-\frac{1}{4}i\pi}}{\sqrt{(2\pi)}} \int_{S(0)} \rho'\{\sin(\psi - \alpha)\} e^{-ikS\cos\alpha} d\alpha. \tag{17}$$

It is now permissible to put $\alpha = 0$ in that part of the integrand which is 'slowly varying' in the vicinity of the saddle-point. When $\psi \doteqdot 0$, the only factor of $\rho'\{\sin(\psi - \alpha)\}$ to which this may not be applicable is that containing the pole at $\psi + \alpha_B$. Hence

$$\Delta(S, \psi) = \frac{e^{-\frac{1}{4}i\pi}}{2\sqrt{(2\pi)}} \sec\tfrac{1}{2}(\psi - \alpha_B)\, \rho''(\sin\psi) \int_{S(0)} \operatorname{cosec}\tfrac{1}{2}(\psi - \alpha + \alpha_B)\, e^{-ikS\cos\alpha} d\alpha, \tag{18}$$

where $$\rho''(\sin\alpha) = -\frac{2}{n}\sqrt{\left(1 - \frac{\cos^2\alpha}{n^2}\right)}(\sin\alpha + \sin\alpha_B) \Big/ \left\{\sin\alpha + \frac{1}{n}\sqrt{\left(1 - \frac{\cos^2\alpha}{n^2}\right)}\right\}. \tag{19}$$

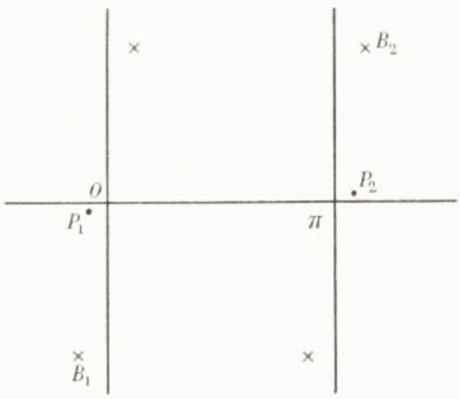

FIGURE 7. Singularities in the complex α-plane: crosses represent branch-points, dots represent poles.

Finally, it can be shown (Clemmow 1951) that the integral in (18) is exactly expressible in terms of the complex Fresnel integral

$$F(a) = e^{ia^2} \int_a^\infty e^{-i\lambda^2} d\lambda, \tag{20}$$

with the result that

$$\Delta(S, \psi) = i\sqrt{2}\sec\tfrac{1}{2}(\psi - \alpha_B)\, \rho''(\sin\psi)\, e^{-ikS} F\{\sqrt{(2kS)}\sin\tfrac{1}{2}(\psi + \alpha_B)\}. \tag{21}$$

This expression is essentially equivalent to the several different forms appearing in the literature, for example, those given by Norton (1941) and Ott (1943). Since we are only concerned with small values of ψ, simplicity is achieved without appreciable loss of accuracy by writing the factor by which the free-space field must be multiplied to give the actual field as

$$A = 1 + \{1 - 4i\gamma_0 F(\gamma)\} e^{-ik(S-R)}, \tag{22}$$

where $$\gamma = \sqrt{(\tfrac{1}{2}kS)}(\sin\psi + \sin\alpha_B), \quad \gamma_0 = \sqrt{(\tfrac{1}{2}kS)}\sin\alpha_B. \tag{23}$$

It will be recognized that $-i\gamma_0^2$ is equivalent to the 'numerical distance' as originally defined by Sommerfeld (1909) for $\psi = 0$, and that $-i\gamma^2$ is effectively the generalized form introduced by Van der Pol & Niessen (1931). Expression (22) is equally applicable to a point-source.

3·2. *Special cases*

Several simple formulae which can be derived from the foregoing analysis will now be given. These help to present a picture of how the field varies, and later we shall look for parallel results in the mixed-path problem.

Our chief concern is with the case when both the transmitter and the receiver are on the earth's surface (a distance d apart); the measured field will then be called the ground-to-ground field, and equation (22) shows that it is given by

$$A = 2\{1-2\mathrm{i}\gamma_0 F(\gamma_0)\} = 2K(\gamma_0), \tag{24}$$

the function $K(a)$ being that introduced in §2·2. When $|\gamma_0| \ll 1$, $A \doteqdot 2$, as though the earth were almost perfectly conducting; whereas, for $|\gamma_0| \gg 1$,

$$A \sim -\frac{\mathrm{i}}{\gamma_0^2} = -\frac{2\mathrm{i}}{kd\sin^2\alpha_B}. \tag{25}$$

The derivation of (25) makes use of the asymptotic expansion of the Fresnel integral, which has, in this case, to be taken to the second term. On the other hand, when the transmitter and/or receiver are sufficiently elevated the first term suffices; applied to (21) it leads to the field of ray theory, which may be written

$$H_z = \frac{\mathrm{e}^{-\mathrm{i}kR}}{\surd(kR)} + \rho(\sin\psi)\frac{\mathrm{e}^{-\mathrm{i}kS}}{\surd(kS)}. \tag{26}$$

This result is, of course, that which would be obtained by removing the complete function $\rho'\{\sin(\psi-\alpha)\}$ from under the integral sign in (17) at the predominant value $\alpha = 0$. The precise conditions under which it gives an adequate representation of the field cannot be put in a simple form, but a useful rough criterion is

$$k(y_0+y)\,|\sin\alpha_B| \gg 1, \tag{27}$$

where y_0 and y are the respective heights of the transmitter and receiver.

Finally, we must introduce the height-gain function. The analysis of §3·1 seems to lead to a more general derivation than that in the literature (e.g. Norton 1941). With some slight transformations, the incident wave (8) may be written

$$H_z^i = \frac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{\surd(2\pi)}\int_{S(0)} \cos\{k(y_0-y)\sin\alpha\}\,\mathrm{e}^{-\mathrm{i}kd\cos\alpha}\,\mathrm{d}\alpha, \tag{28}$$

and the reflected wave (9)

$$H_z^r = \frac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{2\surd(2\pi)}\int_{S(0)} \{[\rho(\sin\alpha)+\rho(-\sin\alpha)]\cos[k(y_0+y)\sin\alpha] \\ -\mathrm{i}[\rho(\sin\alpha)-\rho(-\sin\alpha)]\sin[k(y_0+y)\sin\alpha]\}\,\mathrm{e}^{-\mathrm{i}kd\cos\alpha}\,\mathrm{d}\alpha. \tag{29}$$

Using (6), combination of (28) and (29) gives the total field

$$H_z = \frac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{\surd(2\pi)}\int_{S(0)} \left\{\cos(ky_0\sin\alpha)\cos(ky\sin\alpha) - \frac{1}{n^2}\left(1-\frac{\cos^2\alpha}{n^2}\right)\frac{\sin(ky_0\sin\alpha)\sin(ky\sin\alpha)}{\sin^2\alpha} \right. \\ \left. +\frac{\mathrm{i}}{n}\sqrt{\left(1-\frac{\cos^2\alpha}{n^2}\right)}\frac{\sin[k(y_0+y)\sin\alpha]}{\sin\alpha}\right\}\frac{2\sin^2\alpha}{\sin^2\alpha-\dfrac{1}{n^2}\left(1-\dfrac{\cos^2\alpha}{n^2}\right)}\,\mathrm{e}^{-\mathrm{i}kd\cos\alpha}\,\mathrm{d}\alpha. \tag{30}$$

When it is permissible to put $\alpha = 0$ in that part of the integrand contained in the curly bracket we have

$$H_z = (1 + iky_0 \sin \alpha_B)(1 + iky \sin \alpha_B) H_{z0}, \tag{31}$$

where H_{z0} denotes the ground-to-ground field. Since the height-gain factor in (31) is perhaps most often considered with reference to field-strength only (e.g. Eckersley & Millington 1939; Norton 1941), it may be as well to emphasize that it is equally applicable to phase. It is not easy to judge precisely up to what heights it is valid, but our derivation shows that a sufficient criterion, at least, is that which holds for a perfectly conducting earth ($\sin \alpha_B = 0$), namely,

$$k^2(y_0^2 + y^2) \ll kd. \tag{32}$$

3·3. *Sommerfeld's method*

The method outlined in §3·1 indicates the type of analysis which is used in the sequel. There is, however, an alternative procedure, equivalent to that originally adopted by Sommerfeld (1909), to which it also proves necessary to refer.

The substitution $\cos \alpha = \lambda$ in the integral (9) gives

$$H_z^r = \frac{e^{-\frac{1}{4}i\pi}}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \frac{\rho\{\sqrt{(1-\lambda^2)}\}}{\sqrt{(1-\lambda^2)}} e^{ikd\lambda} e^{-ik(y_0+y)\sqrt{(1-\lambda^2)}} d\lambda. \tag{33}$$

Provided that $\sqrt{(1-\lambda^2)}$ is defined as that branch with a negative imaginary part, the integral in (33) may be evaluated by closing the path of integration with an infinite semicircle above the real axis and appropriate detours round the branch-cuts and poles. It can be shown that the poles of $\rho\{\sqrt{(1-\lambda^2)}\}$ lie in the 'upper' sheet of the Riemann surface, and the singularities, branch-cuts and path of integration appear formally as in figure 8.

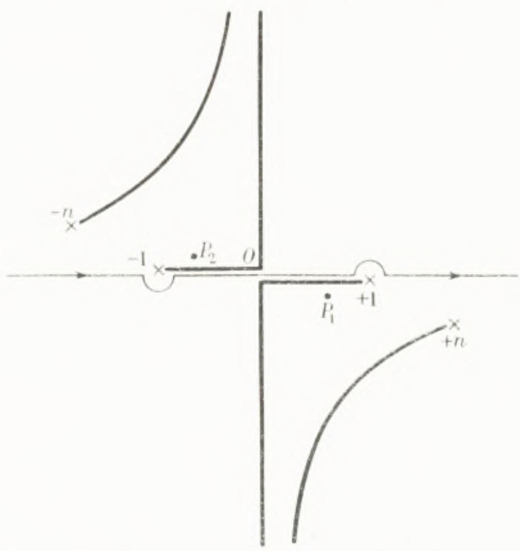

FIGURE 8. Singularities and branch-cuts in the complex λ-plane on Sommerfeld's method.

An appreciation of the positions of the singularities relative to the path of integration is necessary to an understanding of the subsequent analysis, and should be particularly noted in view of a recent suggestion by Epstein (1947) that the latter ought to pass *above* the pole at P_2. In this way Epstein hoped to explain a discrepancy between an expression for the final solution originally given by Sommerfeld and all later versions. This discrepancy (a sign error in the limit of an integral) was first pointed out by Norton (1935) and shown by Burrows (1936, 1937) to be in amount just the 'surface-wave' term contributed by the residue of the pole P_2. Epstein held that the subsequent controversy concerning the 'existence' of this surface wave had never been resolved, a view not unsupported by the text-books (e.g. Stratton 1941, p. 585; Schelkunoff 1943, pp. 430, 431), and his paper inspired a number of others on the same subject (e.g. Kahan & Eckart 1948 *a*, *b*, 1949 *a*, *b*, 1950; further papers have appeared more recently). These have been criticized by Bouwkamp (1948 *a*, *b*, 1950 *a*, *b*, 1951) and it has been established that Epstein's suggestion is incorrect. The essential error (in the present author's opinion) made by Sommerfeld has, however, been overlooked in this revival of an old controversy: namely, that (in the notation of his 1909 paper) he put $\alpha = \sqrt{\rho}$ when α^2 was real and positive, instead of $-\sqrt{\rho}$, as his choice of branch-cuts in fact demanded. This explanation was given by Niessen (1937).

3·4. *Some distinctive features of the analysis*

With reference to the foregoing analysis, it is worth emphasizing several points, an appreciation of which will help to clarify the subsequent work.

(1) A method of solution which is physically straightforward is to express the incident cylindrical field as an angular spectrum of plane waves, choosing a path of integration which is such that the individual plane waves are essentially 'down-coming', thus avoiding any ambiguity in deriving the corresponding reflected field. The resulting integral is conveniently handled by the method of steepest descents.

(2) The mathematics of the problem is characterized by certain poles and branch-points. In distorting the original path of integration to that of steepest descents no pole is ever captured; but a complication arises from the fact that one may lie very close to the saddle-point.

(3) When the earth is homogeneous there is symmetry about the plane through the line-source T and its image T' (figure 5). This symmetry expresses itself in the analysis by the appearance of two relevant pairs of singularities, P_1, B_1 and P_2, B_2 (figures 7, 8). The former come into play when $\psi \doteqdot \pi$, the latter when $\psi \doteqdot 0$.

(4) In Sommerfeld's method of solution a different complex plane of integration is adopted in which the poles P_1 and P_2 appear in the upper sheet of the Riemann surface. This procedure suggests that the residue of P_1 or P_2 contributes explicitly to the field; but such a separation is artificial and only due, as Weyl (1919) was the first to point out, to the rather unnatural mode of attack.

(5) The solution is reciprocal in the sense that it is unaffected by the interchange of transmitter and receiver.

4. The geometric mean formula

In this section we discuss the application to the mixed-path problem of the geometric mean formula (mentioned in §1·2) with particular reference to its limitations.

Suppose that the transmitter and receiver are at equal heights h. From (31), the height-gain function is then

$$(1 + ikh \sin \alpha_B)^2. \tag{34}$$

It is clear from (32) that there could be practical conditions, particularly if kd were very large, under which (34) is valid when

$$kh\,|\sin\alpha_B| \gg 1, \tag{35}$$

and it then becomes effectively

$$-(kh\sin\alpha_B)^2. \tag{36}$$

In these circumstances the ground-to-ground field would be given by (25); an application of (36) therefore yields the corresponding field when the transmitter and the receiver are at equal heights h in the form

$$H_z = \frac{e^{-ikd}}{\sqrt{(kd)}} \frac{2i(kh)^2}{kd}. \tag{37}$$

Now equation (37) has been derived for a homogeneous earth, but is independent of the electrical properties of the ground. Millington (1949*b*) therefore suggests that it should be equally applicable to an inhomogeneous earth, and a reverse use of two height-gain functions, appropriate to the respective media above which the transmitter and receiver are located, then enables him to deduce the ground-to-ground field in this case; the result may be written

$$H_{12z} = \sqrt{(H_{1z} H_{2z})}, \tag{38}$$

where H_{1z} and H_{2z} are the fields pertaining to homogeneous earths composed of the above-mentioned media, and H_{12z} is the field for the composite path. Millington presents the geometric mean formula (38) with explicit reference only to ground-to-ground field-strengths, but it is evidently likewise applicable to the complete field (including phase) at all *equal* heights of the transmitter and receiver up to a maximum determined by the media in question; and, incidentally, the hypothesis of transmission normal to the boundary may be waived.

The fact that a linear differential equation leads to a solution expressed as the geometric mean of two other solutions may appear startling at first sight, but it should be borne in mind that the different fields are all (approximately) proportional to the same inverse power of d, which is the only variable involved, so that (38) is simply a relation between the constants of proportionality.

Millington's recognition of the geometric mean formula plays a considerable part in the development of his technique. In appropriate cases it fixes the mixed-path curve at sufficiently great distances beyond the boundary, and together with the further reasonable assumption that, to a first order, the curve for points up to the boundary coincides with the corresponding curve appropriate to a homogeneous earth, gives some indication of the field-strength variation. The point we wish to make here, however, is that use of equation (38) is only justifiable in special circumstances. The present analysis suggests a criterion for its validity, namely, that the transmitter and receiver should be large numerical distances from the boundary relative to medium 1 and medium 2 respectively, speaking in terms of the model of figure 1; in fact, that this condition is both necessary and sufficient is established more rigorously in part II.

The limitations of equation (38) are perhaps most vividly brought out by noticing that (37) is equivalent to an application of ray theory using an effective reflexion coefficient of -1; for, from (26), this gives the field

$$H_z = \frac{e^{-ikd}}{\sqrt{(kd)}} \{1 - e^{-ik(S-d)}\}; \tag{39}$$

but since (32) must be presumed to hold, we have

$$k(S-d) \doteqdot 2kh^2/d \ll 1, \tag{40}$$

and hence (39) approximates to (37); furthermore, $\rho(\sin\psi) \doteqdot -1$ if $\sin\psi \ll |\sin\alpha_B|$, an inequality which is implicit in (35) and (40). In the problem which we are about to consider, one of the media is a perfect conductor and therefore has a constant reflexion coefficient of $+1$; in this case it appears most forcibly that no argument can be suggested by which the field can be quickly estimated when the transmitter and receiver are on opposite sides of the boundary in positions which are sufficiently near the earth's surface to be of interest, and it seems that convincing results can only be obtained by a thorough analytical investigation. To this we now proceed.

5. The solution for an incident plane wave

This section is devoted to the problem in which the plane wave (4) is incident on the interface depicted in figure 3; the affix p is dropped. The method of solution is precisely that developed elsewhere (Clemmow 1951) in connexion with diffraction problems of a similar type. The currents induced in the diffracting sheet give rise to a *scattered* field which is expressed as an angular spectrum of plane waves, and this representation enables the boundary conditions to be formulated in terms of dual integral equations (Titchmarsh 1937) which can be solved by the use of contour integration.

5·1. *The formulation in terms of dual integral equations*

In the region $y \geqslant 0$, the field of the incident plane wave is

$$\begin{cases} \mathbf{H}^i = (0, 0, 1)\, e^{ikr\cos(\theta-\alpha)}, & (41) \\ \mathbf{E}^i = Z(\sin\alpha, -\cos\alpha, 0)\, e^{ikr\cos(\theta-\alpha)}. & (42) \end{cases}$$

If the perfectly conducting sheet were absent, this would give rise to a reflected wave

$$\begin{cases} \mathbf{H}^r = \rho(\sin\alpha)\,(0, 0, 1)\, e^{ikr\cos(\theta+\alpha)}, & (43) \\ \mathbf{E}^r = Z\rho(\sin\alpha)\,(-\sin\alpha, -\cos\alpha, 0)\, e^{ikr\cos(\theta+\alpha)}, & (44) \end{cases}$$

in the region $y \geqslant 0$, and a transmitted wave

$$\begin{cases} \mathbf{H}^t = \tau(\sin\alpha)\,(0, 0, 1)\, e^{ik'r\cos(\theta-\alpha')}, & (45) \\ \mathbf{E}^t = Z'\tau(\sin\alpha)\,(\sin\alpha', -\cos\alpha', 0)\, e^{ik'r\cos(\theta-\alpha')}, & (46) \end{cases}$$

in the region $y \leqslant 0$; where α' is defined by Snell's law

$$k\cos\alpha = k'\cos\alpha', \tag{47}$$

and the Fresnel reflexion and transmission coefficients are

$$\begin{cases} \rho(\sin\alpha) = \left\{\sin\alpha - \dfrac{1}{n}\sqrt{\left(1-\dfrac{\cos^2\alpha}{n^2}\right)}\right\} \Big/ \left\{\sin\alpha + \dfrac{1}{n}\sqrt{\left(1-\dfrac{\cos^2\alpha}{n^2}\right)}\right\}, & (48) \\ \tau(\sin\alpha) = 2\sin\alpha \Big/ \left\{\sin\alpha + \dfrac{1}{n}\sqrt{\left(1-\dfrac{\cos^2\alpha}{n^2}\right)}\right\}. & (49) \end{cases}$$

When the perfectly conducting sheet is present there will be, in addition to the above fields, a scattered field generated by currents induced in it. We express this scattered field in terms of two angular spectra of plane waves, one for the transmission in free-space, the other for that in the earth. For the half-space $y \geqslant 0$ the non-zero field components may be written as

$$\left\{\begin{aligned} H_z^s &= \int_C P(\cos\beta)\, \mathrm{e}^{-\mathrm{i}kr\cos(\theta-\beta)}\,\mathrm{d}\beta, &(50)\\ E_x^s &= -Z\int_C \sin\beta\, P(\cos\beta)\, \mathrm{e}^{-\mathrm{i}kr\cos(\theta-\beta)}\,\mathrm{d}\beta, &(51)\\ E_y^s &= Z\int_C \cos\beta\, P(\cos\beta)\, \mathrm{e}^{-\mathrm{i}kr\cos(\theta-\beta)}\,\mathrm{d}\beta; &(52)\end{aligned}\right.$$

and for the half-space $y \leqslant 0$ as

$$\left\{\begin{aligned} H_z'^s &= \int_C Q(\cos\beta)\, \mathrm{e}^{-\mathrm{i}k'r\cos(\theta+\beta')}\,\mathrm{d}\beta, &(53)\\ E_x'^s &= Z'\int_C \sin\beta'\, Q(\cos\beta)\, \mathrm{e}^{-\mathrm{i}k'r\cos(\theta+\beta')}\,\mathrm{d}\beta, &(54)\\ E_y'^s &= Z'\int_C \cos\beta'\, Q(\cos\beta)\, \mathrm{e}^{-\mathrm{i}k'r\cos(\theta+\beta')}\,\mathrm{d}\beta. &(55)\end{aligned}\right.$$

A correct behaviour of the scattered field at infinity (outgoing waves) is implicit in these representations. Furthermore, in (53), (54) and (55) β' is some function of β, and $Q(\cos\beta)$ must be expressible in terms of $P(\cos\beta)$. In order to satisfy continuity conditions across $y=0$, β' is clearly given by

$$k\cos\beta = k'\cos\beta', \tag{56}$$

corresponding to (47); and again, the continuity of E_x demands that

$$-Z\sin\beta P(\cos\beta) = Z'\sin\beta' Q(\cos\beta), \tag{57}$$

a relation which reduces correctly to $P(\cos\beta) = -Q(\cos\beta)$ when $n=1$. Substituting from (56) and (57) into (53), (54) and (55), the components of the scattered field in the region $y \leqslant 0$ become

$$\left\{\begin{aligned} H_z'^s &= -\int_C \frac{n\sin\beta}{\sin\beta'} P(\cos\beta)\, \mathrm{e}^{-\mathrm{i}kx\cos\beta+\mathrm{i}k'y\sin\beta'}\,\mathrm{d}\beta, &(58)\\ E_x'^s &= -Z\int_C \sin\beta P(\cos\beta)\, \mathrm{e}^{-\mathrm{i}kx\cos\beta+\mathrm{i}k'y\sin\beta'}\,\mathrm{d}\beta, &(59)\\ E_y'^s &= Z\int_C \sin\beta\cot\beta' P(\cos\beta)\, \mathrm{e}^{-\mathrm{i}kx\cos\beta+\mathrm{i}k'y\sin\beta'}\,\mathrm{d}\beta. &(60)\end{aligned}\right.$$

The complete scattered field is thus expressed in terms of a single angular spectrum function $P(\cos\beta)$.

Now the total field is given by

$$H_z = H_z^i + H_z^r + H_z^s, \quad \text{for } y \geqslant 0, \tag{61}$$

$$H_z' = H_z^t + H_z'^s, \quad \text{for } y \leqslant 0. \tag{62}$$

The boundary conditions which have yet to be satisfied are

(I) $H_z = H_z'$ at $y=0$, $x<0$;

(II) $E_x(=E_x') = 0$ at $y=0$, $x>0$.

But at $y = 0$, $H_z^i + H_z^r = H_z^t$ and $E_x^i + E_x^r = E_x^t$; hence, using (61) and (62), (I) and (II) may be expressed in terms of the unknown scattered field, being respectively replaced by

$$\text{(I}'\text{)} \quad H_z^s = H_z'^s \text{ at } y = 0,\ x < 0;$$

$$\text{(II}'\text{)} \quad E_x^s = -E_x^t \text{ at } y = 0,\ x > 0.$$

If we make the substitution $\cos\beta = \lambda$ in (50), (51), (52) and (58), (59), (60), and also write $\cos\alpha = \lambda_0$, (I′) and (II′) yield a pair of integral equations for $P(\lambda)$, namely,

$$\int_{-\infty}^{\infty} \frac{\sqrt{(1-\lambda^2)} + \frac{1}{n}\sqrt{(1-\lambda^2/n^2)}}{\frac{1}{n}\sqrt{(1-\lambda^2)}\sqrt{(1-\lambda^2/n^2)}} P(\lambda)\, e^{-ikx\lambda}\, d\lambda = 0 \quad \text{for } x < 0, \tag{63}$$

$$\int_{-\infty}^{\infty} P(\lambda)\, e^{-ikx\lambda}\, d\lambda = \frac{\frac{2}{n}\sqrt{(1-\lambda_0^2)}\sqrt{(1-\lambda_0^2/n^2)}}{\sqrt{(1-\lambda_0^2)} + \frac{1}{n}\sqrt{(1-\lambda_0^2/n^2)}}\, e^{ikx\lambda_0} \quad \text{for } x > 0. \tag{64}$$

These are dual integral equations of a type considered elsewhere, and for $n = 1$ they reduce to those arising in the Sommerfeld half-plane diffraction problem (Clemmow 1951). Before solving them it is worth noting several alternative formulations.

5·2. *Alternative formulations*

It has been pointed out in a previous paper (Clemmow 1951) that the use of dual integral equations in certain diffraction problems is an alternative to the use of a single integral equation. The latter method has been developed by Copson (1946*a*, *b*) and a number of American authors, and would be applicable in the present case. For a general solution of equation (63), obtained by taking its Fourier transform, is

$$\frac{\sqrt{(1-\lambda^2)} + \frac{1}{n}\sqrt{(1-\lambda^2/n^2)}}{\frac{1}{n}\sqrt{(1-\lambda^2)}\sqrt{(1-\lambda^2/n^2)}} P(\lambda) = \frac{k}{2\pi}\int_0^{\infty} J_x(\xi)\, e^{ik\lambda\xi}\, d\xi, \tag{65}$$

where $J_x(\xi)$ is an arbitrary function, to be identified, in this application, with the current density in the conducting sheet. If we write formally

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\frac{1}{n}\sqrt{(1-\lambda^2)}\sqrt{(1-\lambda^2/n^2)}}{\sqrt{(1-\lambda^2)} + \frac{1}{n}\sqrt{(1-\lambda^2/n^2)}}\, e^{ik(\xi-x)\lambda}\, d\lambda = \Phi(k\,|\,x-\xi\,|), \tag{66}$$

the substitution of the value of $P(\lambda)$ given by (65) into (64) leads to a single integral equation for $J_x(\xi)$, namely,

$$\tfrac{1}{4}k\int_0^{\infty} J_x(\xi)\, \Phi(k\,|\,x-\xi\,|)\, d\xi = \frac{\frac{2}{n}\sqrt{(1-\lambda_0^2)}\sqrt{(1-\lambda_0^2/n^2)}}{\sqrt{(1-\lambda_0^2)} + \frac{1}{n}\sqrt{(1-\lambda_0^2/n^2)}}\, e^{ikx\lambda_0} \quad \text{for } x > 0. \tag{67}$$

As written, the integral in (66) is not convergent, but its interpretation is quite clear. For consider a current element $J_x(\xi)\,d\xi$, flowing in the x direction, situated in the air-earth interface at $y = 0$, $x = \xi$ (the conducting sheet now being presumed absent); the field to which it gives rise can be obtained by using the method of §3·1 in conjunction with the appropriate angular spectrum, and in the region $y > 0$ the x-component of $\mathbf{E}$ is found to be

$$-\frac{kZ}{4\pi} J_x(\xi)\,d\xi \int_C \{1-\rho(\sin\beta)\} \sin^2\beta\, e^{ikr_1\cos(\theta_1+\beta)}\,d\beta, \tag{68}$$

where r_1 and θ_1 are polar co-ordinates measured from $y = 0$, $x = \xi$. The integral in (68) converges for any given value of θ_1 in the range $0 < \theta_1 < \pi$. Its formal expression at $\theta_1 = 0$ (or π), which is (66), may be defined as the limit when $\theta_1 \to 0$ (or π). Alternatively, convergence at $\theta_1 = 0, \pi$ can be obtained by a permissible distortion of the path C. Thus (68) reduces to

$$-\tfrac{1}{4}kZJ_x(\xi)\,d\xi\,\Phi(k\,|\,x-\xi\,|) \tag{69}$$

on the interface $y = 0$. The corresponding expression for E_x at $y = 0$ due to a current sheet occupying $y = 0$, $x > 0$ is obtained from (69) by integrating over ξ from 0 to ∞. In order to satisfy the boundary conditions on the perfectly conducting plate, this value of E_x must be equated (for $x > 0$) to that of $-E_x^t$ at $y = 0$; the result is the integral equation (67). Equation (67) is of the type susceptible to the method of Wiener & Hopf (Titchmarsh 1937). The Wiener-Hopf procedure would be facilitated by the fact that the kernel $\Phi(k\,|\,x-\xi\,|)$ is defined as a Fourier integral in (66), but this really emphasizes the irrelevance of bringing Φ into the analysis and indicates that the dual integral equations offer a more direct line of attack.

Another slightly different formulation of the problem may be devised. So far we have considered the complete field in terms of a 'correction' to the field existing in the absence of the perfectly conducting sheet. Now let us consider it in terms of a 'correction' to the field which would exist were the conducting sheet infinite instead of semi-infinite. This alternative approach (associated when $n = 1$ with the exact electromagnetic form of Babinet's principle) indeed yields slightly simpler integral equations than those given above, owing to the fact that we are dealing with a vertically polarized field; on the other hand, the new 'correction' field has no obvious physical interpretation. If, then, the whole plane $y = 0$ were occupied by a perfectly conducting sheet, the field in the region $y \geqslant 0$ would consist of the incident wave (41), (42) together with a reflected wave

$$\begin{cases} \mathbf{H}^r = (0, 0, 1)\, e^{ikr\cos(\theta+\alpha)}, & (70)\\ \mathbf{E}^r = Z(-\sin\alpha, -\cos\alpha, 0)\, e^{ikr\cos(\theta+\alpha)}, & (71)\end{cases}$$

and there would be no field in $y \leqslant 0$. When the conducting plate only occupies the area $y = 0$, $x > 0$, there is an additional field which may be cast into the form (50), (51), (52) when $y \geqslant 0$ and into the form (58), (59), (60) when $y \leqslant 0$. The requirement that the resultant E_x for the complete field should be continuous is automatically satisfied, and the boundary conditions which remain to be considered, expressed in a form analogous to (I′) and (II′), are

(I″) $\quad H_z^i + H_z^r + H^s = H_z'^s$ at $y = 0$, $x < 0$;

(II″) $\quad E_x^s (= E_x'^s) = 0$ at $y = 0$, $x > 0$.

These yield the dual integral equations

$$\int_{-\infty}^{\infty} \frac{\sqrt{(1-\lambda^2)}+\frac{1}{n}\sqrt{(1-\lambda^2/n^2)}}{\frac{1}{n}\sqrt{(1-\lambda^2)}\sqrt{(1-\lambda^2/n^2)}} P(\lambda)\,\mathrm{e}^{-\mathrm{i}kx\lambda}\,\mathrm{d}\lambda = -2\,\mathrm{e}^{\mathrm{i}kx\lambda_0} \quad \text{for } x<0, \tag{72}$$

$$\int_{-\infty}^{\infty} P(\lambda)\,\mathrm{e}^{-\mathrm{i}kx\lambda}\,\mathrm{d}\lambda = 0 \quad \text{for } x>0. \tag{73}$$

Again, equations (72) and (73) may be replaced by a single integral equation. The Fourier transform solution of (73) is

$$P(\lambda) = \frac{k}{2\pi}\int_{-\infty}^{0} K_x(\xi)\,\mathrm{e}^{\mathrm{i}k\lambda\xi}\,\mathrm{d}\xi, \tag{74}$$

and substituting for $P(\lambda)$ from (74) into (72) we get

$$k\int_{-\infty}^{0} K_x(\xi)\,\Psi(k\,|\,x-\xi\,|)\,\mathrm{d}\xi = -2\,\mathrm{e}^{\mathrm{i}kx\lambda_0} \quad \text{for } x<0, \tag{75}$$

where

$$\Psi(k\,|\,x-\xi\,|) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sqrt{(1-\lambda^2)}+\frac{1}{n}\sqrt{(1-\lambda^2/n^2)}}{\frac{1}{n}\sqrt{(1-\lambda^2)}\sqrt{(1-\lambda^2/n^2)}}\,\mathrm{e}^{\mathrm{i}k(x-\xi)\lambda}\,\mathrm{d}\lambda. \tag{76}$$

5·3. *The solution*

We now revert to equations (63) and (64), and proceed to solve them by the technique given in a previous paper (Clemmow 1951).

The path of integration is along the real axis except for indentations below the branch-point at $\lambda=-1$ and above that at $\lambda=+1$. A function which is free of singularities and zeros throughout the region above the path of integration, and of algebraic growth at infinity therein, is denoted by U; a function with the same properties below the path of integration by L.

Then a solution of (63) is

$$\frac{\sqrt{(1-\lambda^2)}+\frac{1}{n}\sqrt{(1-\lambda^2/n^2)}}{\frac{2}{n}\sqrt{(1-\lambda^2)}\sqrt{(1-\lambda^2/n^2)}} P(\lambda) = U(\lambda), \tag{77}$$

where the left-hand side has been written in a form which reduces to $P(\lambda)$ when $n=1$. A corresponding solution of (64) is

$$P(\lambda) = -\frac{1}{2\pi\mathrm{i}}\,\frac{\frac{2}{n}\sqrt{(1-\lambda_0^2)}\sqrt{(1-\lambda_0^2/n^2)}}{\sqrt{(1-\lambda_0^2)}+\frac{1}{n}\sqrt{(1-\lambda_0^2/n^2)}}\,\frac{L(\lambda)}{L(-\lambda_0)\,(\lambda+\lambda_0)}, \tag{78}$$

provided that the path of integration is indented *above* the pole at $\lambda=-\lambda_0$.

The elimination of $P(\lambda)$ from (77) and (78) makes it clear that the crux of the problem is the expression of

$$\sqrt{(1-\lambda^2)}+\frac{1}{n}\sqrt{(1-\lambda^2/n^2)} \tag{79}$$

as the product of a U-function and an L-function. The explicit factors could be obtained from the general Wiener-Hopf theory, but they would seem to be too complicated to be of much use here.* However, this difficulty is circumvented in the subsequent analysis, and so we merely write

$$\frac{2}{n}\sqrt{(1-\lambda^2/n^2)}/\{\sqrt{(1-\lambda^2)}+\frac{1}{n}\sqrt{(1-\lambda^2/n^2)}\} = 1/\{U_1(\lambda)\,L_1(\lambda)\}, \tag{80}$$

without, for the moment, inquiring further into the nature of $U_1(\lambda)$ and $L_1(\lambda)$, except to note that $U_1(\lambda) = L_1(-\lambda)$; the particular form of equation (80) has been chosen to make $U_1(\lambda)$ and $L_1(\lambda)$ reduce to unity when $n = 1$. The solution of equations (77) and (78) is now seen to be

$$P(\lambda) = -\frac{1}{2\pi \mathrm{i}}\frac{\sqrt{(1+\lambda_0)}\sqrt{(1+\lambda)}}{L_1(\lambda_0)\,L_1(\lambda)\,(\lambda+\lambda_0)}, \tag{81}$$

where we have applied the result $U_1(\lambda_0) = L_1(-\lambda_0)$. Alternatively,

$$P(\cos\beta) = \frac{\mathrm{i}}{\pi}\frac{1}{L_1(\cos\alpha)\,L_1(\cos\beta)}\frac{\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta}{\cos\alpha+\cos\beta}. \tag{82}$$

At this juncture the opportunity may be taken to interpolate two remarks concerning the nature of our result. First, it is to be noted that (82) reduces to the correct expression for the Sommerfeld half-plane problem when $n = 1$; this check is particularly important in confirming that the solution has the right order of singularity at the origin, as the question of uniqueness is one that demands some attention in diffraction problems (Bouwkamp 1946; Meixner 1949; Copson 1950; Jones 1950; Clemmow 1951). Secondly, we stress the obvious symmetry of (82) in α and β; as will become quite evident shortly, this symmetry is synonymous with the reciprocity criterion, and it is worth convincing oneself that it really demands the factorization expressed by (80). By comparison, we may record the failure, in this respect, of the solution suggested by Raman & Krishnan (1927) for the problem of the diffraction of a plane wave by an imperfectly conducting sheet; the method proposed by Pidduck (1946, 1947) is likewise at fault.

The scattered field is given by (50), (51), (52) or by (58), (59), (60), according as y is positive or negative respectively, with the value (82) for $P(\cos\beta)$. The complete field is then determined by (61) and (62). Thus, for $y \geq 0$,

$$H_z = \mathrm{e}^{\mathrm{i}kr\cos(\theta-\alpha)} + \rho(\sin\alpha)\,\mathrm{e}^{\mathrm{i}kr\cos(\theta+\alpha)} + H_z^s, \tag{83}$$

where
$$H_z^s = \frac{\mathrm{i}}{\pi}\frac{\cos\frac{1}{2}\alpha}{L_1(\cos\alpha)}\int_C \frac{\cos\frac{1}{2}\beta}{L_1(\cos\beta)\,(\cos\beta+\cos\alpha)}\,\mathrm{e}^{-\mathrm{i}kr\cos(\theta-\beta)}\,\mathrm{d}\beta; \tag{84}$$

and for $y \leq 0$
$$H_z' = \tau(\sin\alpha)\,\mathrm{e}^{\mathrm{i}k'r\cos(\theta-\alpha')} + H_z'^s, \tag{85}$$

where
$$H_z'^s = -\frac{\mathrm{i}}{\pi}\frac{n\cos\frac{1}{2}\alpha}{L_1(\cos\alpha)}\int_C \frac{\sin\beta\cos\frac{1}{2}\beta}{\sin\beta'\,L_1(\cos\beta)\,(\cos\beta+\cos\alpha)}\,\mathrm{e}^{-\mathrm{i}k'r\cos(\theta+\beta')}\,\mathrm{d}\beta. \tag{86}$$

The corresponding expressions for E_x, E_y, and E_x', E_y' may be written down in a like manner.

* Senior (1952), in considering the rather similar problem of diffraction by an imperfectly conducting half-plane, has in effect worked out the 'split' for $1/n+\sqrt{(1-\lambda^2)}$ which is a valid approximation to (79) when $|n| \gg 1$ (see §12·2).

5·4. *A transformation of the solution*

In the propagation problem the field below the earth's surface is generally of no concern; the subsequent discussion is therefore confined to the region $y \geqslant 0$.

Following a common practice in diffraction theory, our first aim is to separate the expression (83) for H_z into the sum of a geometrical optics term and a diffraction term, as discussed in §2·1. To this end, the path C in (84) must be distorted into $S(\theta)$, the path of steepest descents; a knowledge of the nature and location of the singularities of the integrand, in particular of $1/L_1(\cos\beta)$, is therefore necessary.

Since the functions U_1 and L_1 are defined by (80) in terms of λ, it is desirable, for the moment, to revert to the complex λ-plane. Referring to §3·1, equation (16), the poles of (80) are given by

$$\lambda = \pm n/\surd(1+n^2), \tag{87}$$

the upper and lower signs corresponding to P_1 and P_2 respectively in figure 7. We are now confronted with a slight difficulty. If in the complex λ-plane we adopt the branch-cuts appropriate to the relations $\lambda = \cos\beta$, $\surd(1-\lambda^2) = \sin\beta$, as shown in figure 9 ($\surd(1-\lambda^2)$ positive real part), the poles given by (87) do not appear in the upper sheet of the Riemann surface; this is evident because the upper sheet in the λ-plane then maps into the region $0 \leqslant \mathscr{R}\beta \leqslant \pi$ in the β-plane, and P_1 and P_2 lie outside this region (figure 7). On the other hand, in order to determine which pole belongs to $1/U_1(\lambda)$ and which to $1/L_1(\lambda)$, it seems necessary to bring them into the upper sheet of the Riemann surface; this is achieved, as indicated in §3·3, by introducing the branch-cuts shown in figure 8. For the moment, therefore, we must think in terms of the technique of closing the path of integration with an infinite semicircle (corresponding to Sommerfeld's original procedure for the homogeneous earth analysis), although this is not the most suitable approach, and not the one which we shall eventually use; it is then clear that the pole P_1 belongs to $1/U_1(\lambda)$ and the pole P_2 to $1/L_1(\lambda)$. With regard to the branch-points of (80) the matter is of course quite straightforward; the branch-points at $+1$, $+n$ belong to $U_1(\lambda)$ and those at -1, $-n$ to $L_1(\lambda)$.

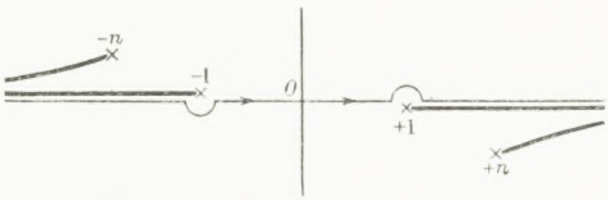

FIGURE 9. Singularities and branch-cuts in the complex λ-plane on the present method.

Having established the above results, we revert once again to the complex β-plane. Referring to figure 7, the relevant singularities of $1/L_1(\cos\beta)$ are the pole at P_2 and the branch-point at B_2. In distorting the path C into the path $S(\theta)$, the pole at P_2 will never be captured, but the branch-point at B_2 will be crossed when θ is sufficiently large. If the present analysis be compared with that previously given for a homogeneous earth, it will be remarked that the singularities at P_2 and B_2 play much the same role as before, but that, in contrast, there are no longer any singularities at P_1 and B_1. This is what might have been expected from simple physical considerations. For a homogeneous earth, the presence of singularities at P_1 and B_1, in addition to those at P_2 and B_2, corresponds to the symmetry of the configuration; in the mixed-path problem, however, this symmetry has disappeared. The effect of the lack of symmetry can, indeed, be brought out more explicitly by a crude interpretation of the diffraction field as some sort of 'edge-wave' emanating from the discontinuity at the boundary. When θ is nearly equal to π the field of this edge-wave has been transmitted over an imperfectly conducting region of the earth's surface, and we must therefore expect features corresponding to the homogeneous earth analysis of § 3·1 to present themselves; they do, in the guise of the distinctive singularities at P_2 and B_2. On the other hand, when θ is nearly zero, the field of the edge-wave has been propagated over a surface of infinite conductivity, and consequently it is equally to be expected that the analysis should not be appreciably affected by any singularities.

It has been shown that the distortion of the path C to the path $S(\theta)$ will, in certain circumstances, capture the branch-point at B_2, and an appropriate branch-cut integral should then be included in the rigorous solution; it is, however, legitimate to neglect this contribution, the justification for such a procedure resting on essentially the same argument as that suggested in the corresponding stage of the analysis for the case of a homogeneous earth. On the other hand, the poles of the integrand of (84) given by $\cos\beta+\cos\alpha=0$ must be considered; these poles play the same part in the analysis as they do in the simple diffraction problem to which the present problem reduces when $n=1$; since the case when $n=1$ has been treated elsewhere by this method (Clemmow 1951) we need only note here that the residue of the integrand at $\beta=\pi-\alpha$ would, if the pole were encircled positively, contribute the term

$$\frac{e^{ikr\cos(\theta+\alpha)}}{L_1(\cos\alpha)\,L_1(-\cos\alpha)}=\{1-\rho(\sin\alpha)\}\,e^{ikr\cos(\theta+\alpha)}. \tag{88}$$

It is then apparent, from an examination of the different cases, that the field in the region $y\geq 0$, given by (83) and (84), may be written

$$H_z=e^{ikr\cos(\theta-\alpha)}+\binom{1}{\rho}e^{ikr\cos(\theta+\alpha)}+H_z^d, \tag{89}$$

where

$$H_z^d=\frac{i}{\pi}\frac{\cos\frac{1}{2}\alpha}{L_1(\cos\alpha)}\int_{S(\theta)}\frac{\cos\frac{1}{2}\beta}{L_1(\cos\beta)\,(\cos\beta+\cos\alpha)}\,e^{-ikr\cos(\theta-\beta)}\,d\beta, \tag{90}$$

and

$$\binom{1}{\rho}=\begin{cases}1 & \text{for } 0\leq\theta\leq\pi-\alpha,\\ \rho(\sin\alpha) & \text{for } \pi-\alpha\leq\theta\leq\pi.\end{cases} \tag{91}$$

The first two terms of (89) give the expected field of geometrical optics, and (90) is the corresponding diffraction field.

As far as we are here concerned, the solution associated with an incident plane wave merely serves as a link in the analytical chain, and it is not accorded an independent development. We need only remark that the integral equation approach, in contrast to that attempted by Hanson (1938), yields the answer in a compact form which is particularly suitable for conversion to that appropriate to a line-source. This procedure is carried out in the next section.

6. The solution for a line-source

6·1. *The general form*

In the model of figure 3 we consider a line-source T situated at (r_0, θ_0) which would, in free-space, propagate the cylindrical wave (3). Again we express this primary field as the angular spectrum of plane waves (8), and the complete field is therefore obtained on multiplying (89) by the factor

$$\frac{e^{-\frac{1}{4}i\pi}}{\sqrt{(2\pi)}} e^{-ikr_0\cos(\theta_0-\alpha)}, \tag{92}$$

and integrating with respect to α over a suitable path. It now proves convenient to take the path as $S(\frac{1}{2}\pi)$ rather than C, and the result is

$$H_z = \frac{e^{-\frac{1}{4}i\pi}}{\sqrt{(2\pi)}} \int_{S(\frac{1}{2}\pi)} \left\{ e^{ikr\cos(\theta-\alpha)} + \binom{1}{\rho} e^{ikr\cos(\theta+\alpha)} \right\} e^{-ikr_0\cos(\theta_0-\alpha)} d\alpha + \frac{e^{-\frac{1}{4}i\pi}}{\sqrt{(2\pi)}} \int_{S(\frac{1}{2}\pi)} H_z^{pd} e^{-ikr_0\cos(\theta_0-\alpha)} d\alpha, \tag{93}$$

where H_z^{pd} is given by (90), the superscript p being reintroduced to distinguish the field associated with an incident plane wave.

The next step is to express (93) in turn as the sum of a geometrical optics term and a diffraction term; to which end the path of integration for α in the second integral of (93) must be displaced to that of steepest descents, $S(\theta_0)$. This procedure is natural from considerations of symmetry, and is identical with that demonstrated elsewhere (Clemmow 1950*c*) for the simpler case when $n = 1$.

In displacing the path we must take into account the singularities of H_z^{pd} regarded as a function of α. First, there are the singularities belonging to $1/L_1(\cos\alpha)$; the poles of this function lie outside the region between $S(0)$ and $S(\pi)$, and hence are not captured; a branch-point may be crossed, but again the associated branch-cut integral is permissibly neglected (indeed, of necessity, to keep the approximations consistent). Secondly, the integrand of (90) has poles in the complex α-plane at $\cos\alpha = -\cos\beta$; H_z^{pd} therefore has poles given by this relation where β assumes all values on the path $S(\theta)$, and it is the contribution of their residues which combines with the first integral in (93) to yield the geometrical optics term of the solution. Indeed, the residue of

$$\frac{i}{\pi} \frac{\cos\frac{1}{2}\alpha}{L_1(\cos\alpha)} \frac{\cos\frac{1}{2}\beta}{L_1(\cos\beta)(\cos\beta+\cos\alpha)} e^{-ikr_0\cos(\theta_0-\alpha)} \tag{94}$$

at the pole $\alpha = \pi - \beta$ is

$$\frac{1}{i\pi} \frac{\sin\frac{1}{2}\beta\cos\frac{1}{2}\beta}{L_1(-\cos\beta)\,L_1(\cos\beta)\sin\beta} e^{ikr_0\cos(\theta_0+\beta)} = \frac{1}{2\pi i}\{1-\rho(\sin\beta)\} e^{ikr_0\cos(\theta_0+\beta)}. \tag{95}$$

Thus we may write (93) in the form

$$H_z = H_z^g + H_z^d, \tag{96}$$

where

$$H_z^g = \begin{cases} \sqrt{(\frac{1}{2}\pi)}\, e^{-\frac{1}{4}i\pi}\{H_0^{(2)}(kR) + H_0^{(2)}(kS)\} & \text{for } 0 \leqslant \theta \leqslant \pi-\theta_0, \\ \sqrt{(\frac{1}{2}\pi)}\, e^{-\frac{1}{4}i\pi} H_0^{(2)}(kR) + \dfrac{e^{-\frac{1}{4}i\pi}}{\sqrt{(2\pi)}} \displaystyle\int_{S(\frac{1}{2}\pi)} \rho(\sin\alpha)\, e^{ikS\cos(\psi+\alpha)} d\alpha & \text{for } \pi-\theta_0 \leqslant \theta \leqslant \pi, \end{cases} \tag{97}$$

and

$$H_z^d = \frac{e^{\frac{1}{4}i\pi}}{\pi\sqrt{(2\pi)}} \int_{S(\theta_0)} \int_{S(\theta)} \frac{\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta}{L_1(\cos\alpha)\,L_1(\cos\beta)\,(\cos\alpha+\cos\beta)} e^{-ik\{r_0\cos(\theta_0-\alpha)+r\cos(\theta-\beta)\}} d\beta\, d\alpha. \tag{98}$$

The geometrical optics term (97) conforms to expectation, the two expressions involved being the respective fields appropriate to a perfectly conducting earth and an imperfectly conducting earth with a Fresnel reflexion coefficient $\rho(\sin\alpha)$. If we introduce the notation of § 3·1, (97) appears in the more compact form

$$H_z^g = \sqrt{(\tfrac{1}{2}\pi)}\, e^{-\frac{1}{4}i\pi}\{H_0^{(2)}(kR)+H_0^{(2)}(kS)\}+\binom{0}{1}\Delta(S,\psi), \tag{99}$$

where

$$\binom{0}{1}=\begin{cases}0 & \text{for } 0\leqslant\theta\leqslant\pi-\theta_0,\\ 1 & \text{for } \pi-\theta_0\leqslant\theta\leqslant\pi.\end{cases} \tag{100}$$

It will be observed that the solution is reciprocal in the sense that it is unaltered by an interchange of r_0, θ_0 and r, θ.

6·2. *A simplification*

The essential complication of the mixed-path problem lies in the evaluation of the double integral (98) for H_z^d; this is the diffraction field which smooths out the discontinuity in the geometrical optics field (99), and is obviously of major importance in the cases of practical interest for which $\theta+\theta_0$ is near π. The immediate obstacle to progress is that no reasonable expression for $L_1(\cos\alpha)$ is available. We can, however, introduce an initial simplification by adopting the powerful arguments associated with the method of integration by steepest descents, and shall shortly see how this resolves the difficulty.

The predominant values of α and β in (98) are θ_0 and θ respectively; hence, if kr and kr_0 are large it should be permissible to put $\alpha=\theta_0$, $\beta=\theta$ in those parts of the integrand of (98) which are 'slowly varying' in the neighbourhood of these values. As far as the author is aware a rigorous mathematical treatment of this process applied to a *double* integral has yet to be given, but the required extension of the standard justification in the case of a single integral appears sufficiently straightforward to warrant no hesitation in its use.

In preparation for this procedure we write

$$U_1(\cos\alpha)\,L_1(\cos\alpha) = U_2(\cos\alpha)\,L_2(\cos\alpha)\sin\tfrac{1}{2}(\alpha+\alpha_B)\cos\tfrac{1}{2}(\alpha-\alpha_B), \tag{101}$$

where, from (19),

$$U_2(\cos\alpha)\,L_2(\cos\alpha) = -2/\rho''(\sin\alpha). \tag{102}$$

Now the pole P_1 is given by $\sin\frac{1}{2}(\alpha+\alpha_B)=0$ and the pole P_2 by $\cos\frac{1}{2}(\alpha-\alpha_B)=0$ (see figure 7); hence the single equation (101) implies the pair of equations

$$U_1(\cos\alpha) = U_2(\cos\alpha)\sin\tfrac{1}{2}(\alpha+\alpha_B), \tag{103}$$

$$L_1(\cos\alpha) = L_2(\cos\alpha)\cos\tfrac{1}{2}(\alpha-\alpha_B). \tag{104}$$

Furthermore, the only singularities of (102) are the branch-points at $\cos\alpha=+n$ and $\cos\alpha=-n$, the former belonging to $1/U_2(\cos\alpha)$ and the latter to $1/L_2(\cos\alpha)$; thus, both $1/U_2(\cos\alpha)$ and $1/L_2(\cos\alpha)$ are 'slowly varying' for values of α near 0 or π.

It is therefore reasonable to suppose that an adequate approximation to (98) is

$$H_z^d = \frac{e^{\frac{1}{4}i\pi}}{\pi\sqrt{(2\pi)}}\frac{1}{L_2(\cos\theta_0)\,L_2(\cos\theta)} \times\int_{S(\theta_0)}\int_{S(\theta)}\frac{\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta}{\cos\frac{1}{2}(\alpha-\alpha_B)\cos\frac{1}{2}(\beta-\alpha_B)(\cos\alpha+\cos\beta)}\,e^{-ik\{r_0\cos(\theta_0-\alpha)+r\cos(\theta-\beta)\}}\,d\beta\,d\alpha. \tag{105}$$

This is conveniently written in the form

$$H_z^d = \frac{e^{\frac{1}{4}i\pi}}{4\pi\sqrt{(2\pi)}}\frac{1}{L_2(\cos\theta_0)\,L_2(\cos\theta)}\,(H_z^{d1}+H_z^{d2}), \tag{106}$$

where

$$H_z^{d1} = \int_{S(0)}\int_{S(0)} \sec\tfrac{1}{2}(\alpha+\theta_0-\alpha_B)\sec\tfrac{1}{2}(\beta+\theta-\alpha_B)\sec\tfrac{1}{2}(\alpha-\beta+\theta_0-\theta)\,e^{-ik(r_0\cos\alpha+r\cos\beta)}\,d\alpha\,d\beta, \tag{107}$$

$$H_z^{d2} = \int_{S(0)}\int_{S(0)} \sec\tfrac{1}{2}(\alpha+\theta_0-\alpha_B)\sec\tfrac{1}{2}(\beta+\theta-\alpha_B)\sec\tfrac{1}{2}(\alpha+\beta+\theta_0+\theta)\,e^{-ik(r_0\cos\alpha+r\cos\beta)}\,d\alpha\,d\beta. \tag{108}$$

The main task of the next section is to express the double integrals (107) and (108) in terms of single integrals which are suitable for computation. But even when this is achieved, the solution still requires, as (106) shows, the evaluation of

$$L_2(\cos\theta_0)\,L_2(\cos\theta). \tag{109}$$

For general values of θ and θ_0 this would be a tedious process. However, if $\theta+\theta_0=\pi$, (109) becomes

$$L_2(\cos\theta_0)\,L_2(-\cos\theta_0) = L_2(\cos\theta_0)\,U_2(\cos\theta_0), \tag{110}$$

which is easily calculated from (102). The condition $\theta+\theta_0=\pi$ caters for our chief interest, which is in the ground-to-ground field when the transmitter and receiver are on opposite sides of the boundary; moreover, it allows us to check the validity of the height-gain analysis in the mixed-path problem, which may therefore be used to some extent to derive the field for an elevated transmitter and receiver.

7. The reduction of the solution

7·1. *The reduction of H_z^{d2} when $\theta \doteqdot 0$, $\theta_0 \doteqdot \pi$*

The discontinuity in the diffraction field across OI (figure 4) arises from the expression H_z^{d2} whose reduction we consider first. We start by treating the case for which θ_0 is just less than π and θ just greater than 0, so that the transmitter is situated over the imperfectly conducting ground and the receiver over the perfect conductor; since the answer is strictly reciprocal the results are immediately applicable to the case $\theta_0 \doteqdot 0$, $\theta \doteqdot \pi$. It may therefore be assumed that $|\theta-\alpha_B|$ is small; consequently in (108) the poles of the integrand given by $\cos\frac{1}{2}(\beta+\theta-\alpha_B)=0$ are not near the predominant value $\beta=0$. Thus, it is permissible to write

$$H_z^{d2} = \sec\tfrac{1}{2}(\theta-\alpha_B)\int_{S(0)}\int_{S(0)} \sec\tfrac{1}{2}(\alpha+\theta_0-\alpha_B)\sec\tfrac{1}{2}(\alpha+\beta+\theta_0+\theta)\,e^{-ik(r_0\cos\alpha+r\cos\beta)}\,d\alpha\,d\beta. \tag{111}$$

In (111) make the steepest descents substitutions

$$\xi = \sqrt{2}\,e^{-\frac{1}{4}i\pi}\sin\tfrac{1}{2}\alpha, \quad \eta = \sqrt{2}\,e^{-\frac{1}{4}i\pi}\sin\tfrac{1}{2}\beta, \tag{112}$$

and neglect, where appropriate, ξ^2, η^2, $\xi\eta$, and higher-order terms in ξ and η. This gives

$$H_z^{d2} = \frac{4\,e^{-ikR_1}}{\cos\frac{1}{2}(\theta-\alpha_B)\sin\frac{1}{2}(\theta_0-\alpha_B)\sin\frac{1}{2}(\theta+\theta_0)} \times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-k(r_0\xi^2+r\eta^2)}}{\{\xi-\sqrt{2}\,e^{-\frac{1}{4}i\pi}\cot\frac{1}{2}(\theta_0-\alpha_B)\}\{\xi+\eta-\sqrt{2}\,e^{-\frac{1}{4}i\pi}\cot\frac{1}{2}(\theta+\theta_0)\}}\,d\xi\,d\eta, \tag{113}$$

where $R_1 = r_0 + r$. Next, make the polar substitutions

$$\xi = \surd(R_1/r_0)\cos\phi, \quad \eta = \surd(R_1/r)\sin\phi. \tag{114}$$

Then

$$H_z^{d2} = 4\surd(r_0/R_1)\,\mathrm{e}^{-\mathrm{i}kR_1}\sec\tfrac{1}{2}(\theta-\alpha_B)\,\mathrm{cosec}\,\tfrac{1}{2}(\theta_0-\alpha_B)\,\mathrm{cosec}\,\tfrac{1}{2}(\theta+\theta_0)\int_0^\infty \rho J(\rho)\,\mathrm{e}^{-kR_1\rho^2}\,\mathrm{d}\rho, \tag{115}$$

where $$J(\rho) = \int_0^{2\pi}\left\{\rho\cos\phi - \mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\sqrt{\left(\frac{2r_0}{R_1}\right)}\cot\tfrac{1}{2}(\theta_0-\alpha_B)\right\}^{-1}$$

$$\times\left\{\rho\left(\sqrt{\frac{r}{R_1}}\cos\phi + \sqrt{\frac{r_0}{R_1}}\sin\phi\right) - \mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\frac{\surd(2rr_0)}{R_1}\cot\tfrac{1}{2}(\theta+\theta_0)\right\}^{-1}\mathrm{d}\phi. \tag{116}$$

$J(\rho)$ can be evaluated by a standard technique, using the substitution $z = \exp(\mathrm{i}\phi)$. This substitution gives

$$J(\rho) = -\frac{4\mathrm{i}}{\rho^2}\int_{\substack{\text{unit}\\\text{circle}}}\frac{z\,\mathrm{d}z}{(z^2-2az/\rho+1)\,(Az^2-2Bz/\rho+C)}, \tag{117}$$

where
$$\left.\begin{aligned} a &= \mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\sqrt{\left(\frac{2r_0}{R_1}\right)}\cot\tfrac{1}{2}(\theta_0-\alpha_B),\\ A &= \surd(r/R_1)-\mathrm{i}\surd(r_0/R_1), \quad C = \surd(r/R_1)+\mathrm{i}\surd(r_0/R_1),\\ B &= \mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\frac{\surd(2rr_0)}{R_1}\cot\tfrac{1}{2}(\theta+\theta_0). \end{aligned}\right\} \tag{118}$$

The poles of the integrand of (117) are

$$z_1 = \{a+\mathrm{i}\surd(\rho^2-a^2)\}/\rho, \quad z_2 = \{a-\mathrm{i}\surd(\rho^2-a^2)\}/\rho, \tag{119}$$

and
$$Z_1 = \frac{1}{\rho}\sqrt{\frac{C}{A}}\left\{\frac{B}{\surd(AC)}+\mathrm{i}\sqrt{\left(\rho^2-\frac{B^2}{AC}\right)}\right\}, \quad Z_2 = \frac{1}{\rho}\sqrt{\frac{C}{A}}\left\{\frac{B}{\surd(AC)}-\mathrm{i}\sqrt{\left(\rho^2-\frac{B^2}{AC}\right)}\right\}. \tag{120}$$

Since $z_1 z_2 = 1$ and $|Z_1 Z_2| = 1$, one and only one of each of the pairs (119) and (120) lies within the unit circle. If $\surd(\rho^2-a^2)$ and $\surd\{\rho^2-B^2/(AC)\}$ are defined as those branches with positive real parts, it is not difficult to show (when ρ is real) that the poles within the unit circle are z_1, Z_1 for $\theta+\theta_0<\pi$, and z_1, Z_2 for $\theta+\theta_0>\pi$. Now write

$$J(\rho) = -\mathrm{i}\int_{\substack{\text{unit}\\\text{circle}}}\left(\frac{p_1}{z-z_1}+\frac{p_2}{z-z_2}+\frac{P_1}{z-Z_1}+\frac{P_2}{z-Z_2}\right)\mathrm{d}z, \tag{121}$$

so that
$$J(\rho) = \begin{cases} 2\pi(p_1+P_1) & \text{for } \theta+\theta_0<\pi, \quad (122)\\ 2\pi(p_1+P_2) & \text{for } \theta+\theta_0>\pi. \quad (123)\end{cases}$$

The following results can be obtained:

$$p_1 = -\mathrm{i}\surd(R_1/r_0)/\{\surd(\rho^2-a^2)\,[\surd(\rho^2-a^2)+a\surd(r/r_0)-B\surd(R_1/r_0)]\}, \tag{124}$$

$$P_1 = \mathrm{i}\surd(R_1/r_0)/\{\surd(\rho^2-B^2)\,[\surd(\rho^2-B^2)+a\surd(R_1/r_0)-B\surd(r/r_0)]\}, \tag{125}$$

$$P_2 = \mathrm{i}\surd(R_1/r_0)/\{\surd(\rho^2-B^2)\,[\surd(\rho^2-B^2)-a\surd(R_1/r_0)+B\surd(r/r_0)]\}. \tag{126}$$

Hence, from (122) and (123), we have

$$J(\rho) = 2\pi\mathrm{i}\sqrt{\frac{R_1}{r_0}}\left\{(\rho^2-B^2)^{-\frac{1}{2}}\left[\sqrt{(\rho^2-B^2)}\pm a\sqrt{\frac{R_1}{r_0}}\mp B\sqrt{\frac{r}{r_0}}\right]^{-1} -(\rho^2-a^2)^{-\frac{1}{2}}\left[\sqrt{(\rho^2-a^2)}+a\sqrt{\frac{r}{r_0}}-B\sqrt{\frac{R_1}{r_0}}\right]^{-1}\right\}, \quad (127)$$

with the upper sign for $\theta+\theta_0<\pi$ and the lower sign for $\theta+\theta_0>\pi$.

From (115), therefore,

$$H_z^{d2} = 8\pi\mathrm{i}\,\mathrm{e}^{-\mathrm{i}kR_1}\sec\tfrac{1}{2}(\theta-\alpha_B)\,\mathrm{cosec}\,\tfrac{1}{2}(\theta_0-\alpha_B)\,\mathrm{cosec}\,\tfrac{1}{2}(\theta+\theta_0)\,I, \quad (128)$$

where

$$I = \int_0^\infty \frac{\rho\,\mathrm{e}^{-kR_1\rho^2}\,\mathrm{d}\rho}{\sqrt{(\rho^2-B^2)}\left\{\sqrt{(\rho^2-B^2)}\pm a\sqrt{\frac{R_1}{r_0}}\mp B\sqrt{\frac{r}{r_0}}\right\}} - \int_0^\infty \frac{\rho\,\mathrm{e}^{-kR_1\rho^2}\,\mathrm{d}\rho}{\sqrt{(\rho^2-a^2)}\left\{\sqrt{(\rho^2-a^2)}+a\sqrt{\frac{r}{r_0}}-B\sqrt{\frac{R_1}{r_0}}\right\}}, \quad (129)$$

with the upper sign for $\theta+\theta_0<\pi$ and the lower sign for $\theta+\theta_0>\pi$. In the first and second integrals of (129) make the respective substitutions

$$\lambda = \sqrt{(\rho^2-B^2)}, \quad \lambda = \sqrt{(\rho^2-a^2)}; \quad (130)$$

then

$$I = \mathrm{e}^{-kR_1B^2}\int_{\pm\mathrm{i}B}^\infty \frac{\mathrm{e}^{-kR_1\lambda^2}\,\mathrm{d}\lambda}{\lambda\pm a\sqrt{\frac{R_1}{r_0}}\mp B\sqrt{\frac{r}{r_0}}} - \mathrm{e}^{-kR_1a^2}\int_{\mathrm{i}a}^\infty \frac{\mathrm{e}^{-kR_1\lambda^2}\,\mathrm{d}\lambda}{\lambda+a\sqrt{\frac{r}{r_0}}-B\sqrt{\frac{R_1}{r_0}}}, \quad (131)$$

with the upper sign for $\theta+\theta_0<\pi$ and the lower sign for $\theta+\theta_0>\pi$. The individual integrals in (131) have the unpleasant feature that they diverge if $a=B=0$, but I itself does not, as the following analysis shows. Consider the transformation

$$\mathrm{e}^{-kR_1\alpha^2}\int_{\mathrm{i}\alpha}^\infty \frac{\mathrm{e}^{-kR_1\lambda^2}}{\lambda+\beta}\,\mathrm{d}\lambda = \mathrm{e}^{-kR_1(\alpha^2+\beta^2)}\int_{\mathrm{i}\sqrt{(\alpha^2+\beta^2)}}^\infty \frac{\mathrm{e}^{-kR_1\lambda^2}}{\lambda}\,\mathrm{d}\lambda - \beta\,\mathrm{e}^{-kR_1\alpha^2}\int_{\mathrm{i}\alpha}^\infty \frac{\mathrm{e}^{-kR_1\lambda^2}}{\lambda^2-\beta^2}\,\mathrm{d}\lambda. \quad (132)$$

The second term on the right-hand side of (132) is finite at $\alpha=\beta=0$; the first term is not, but it depends on α and β only via the combination $\alpha^2+\beta^2$. Applying the transformation (132) to each expression in (131) the two contributions which diverge for $a=B=0$ cancel out, and so

$$I = \mp\left(a\sqrt{\frac{R_1}{r_0}}-B\sqrt{\frac{r}{r_0}}\right)\mathrm{e}^{-kR_1B^2}\int_{\pm\mathrm{i}B}^\infty \frac{\mathrm{e}^{-kR_1\lambda^2}\,\mathrm{d}\lambda}{\lambda^2-\left(a\sqrt{\frac{R_1}{r_0}}-B\sqrt{\frac{r}{r_0}}\right)^2} + \left(a\sqrt{\frac{r}{r_0}}-B\sqrt{\frac{R_1}{r_0}}\right)\mathrm{e}^{-kR_1a^2}\int_{\mathrm{i}a}^\infty \frac{\mathrm{e}^{-kR_1\lambda^2}\,\mathrm{d}\lambda}{\lambda^2-\left(a\sqrt{\frac{r}{r_0}}-B\sqrt{\frac{R_1}{r_0}}\right)^2}, \quad (133)$$

with the upper sign for $\theta+\theta_0<\pi$ and the lower sign for $\theta+\theta_0>\pi$.

H_z^{d2} is given by (128) and (133) in essentially the reduced form which we have been seeking.

7·2. *The reduction of H_z^{d1} when $\theta \doteqdot 0$, $\theta_0 \doteqdot \pi$*

An expression for H_z^{d1} in terms of H_z^{d2} is easily obtained. From (107), using the approximation corresponding to (111) and substituting $-\beta$ for β, we have

$$H_z^{d1} = \sec\tfrac{1}{2}(\theta-\alpha_B)\int_{S(0)}\int_{S(0)}\sec\tfrac{1}{2}(\alpha+\theta_0-\alpha_B)\sec\tfrac{1}{2}(\alpha+\beta+\theta_0-\theta)\,\mathrm{e}^{-\mathrm{i}k(r_0\cos\alpha+r\cos\beta)}\,\mathrm{d}\alpha\,\mathrm{d}\beta. \quad (134)$$

A comparison of (134) with (111) shows that

$$\cos\tfrac{1}{2}(\theta-\alpha_B)\,H_z^{d1}(r_0,r,\theta_0,\theta)=\cos\tfrac{1}{2}(\theta+\alpha_B)\,H_z^{d2}(r_0,r,\theta_0,-\theta). \tag{135}$$

The interpretation of (135) requires a little care; for $H_z^{d2}(r_0,r,\theta_0,\theta)$ is represented by different functions in the two cases $\theta_0+\theta<\pi$, $\theta_0+\theta>\pi$, whereas H_z^{d1} is continuous for all values of θ_0 and θ between 0 and π. Since $\theta_0-\theta<\pi$, (135) holds if $H_z^{d2}(r_0,r,\theta_0,-\theta)$ is obtained by substituting $-\theta$ for θ in the expression for $H_2^{d2}\,(r_0,r,\theta_0,\theta)$ appropriate to the condition $\theta_0+\theta<\pi$ (upper sign in equation (133)).

7·3. *The case* $\theta \doteqdot \pi$, $\theta_0 \doteqdot \pi$

The case when the receiver is on the same side of the boundary as the transmitter is very quickly dealt with. For points well away from the lines $\theta+\theta_0=\pi$ and $\theta-\theta_0=\pi$ it is permissible, to the required order of approximation, to put $\alpha=\beta=0$ in the factors $\sec\frac{1}{2}(\alpha-\beta+\theta_0-\theta)$ and $\sec\frac{1}{2}(\alpha+\beta+\theta_0+\theta)$ in the respective integrands of (107) and (108). The procedure is comparable with that in ordinary diffraction theory ($n=1$) which leads to the edge-wave approximation for the diffraction field, and which, for a primary line-source, is valid in a region outside two hyperbolas whose axes are $\theta+\theta_0=\pi$ and $\theta-\theta_0=\pi$ (Clemmow 1950*c*). It gives

$$H_z^{d1}+H_z^{d2}=\frac{4\cos\frac{1}{2}\theta_0\cos\frac{1}{2}\theta}{\cos\theta_0+\cos\theta}\int_{S(0)}\int_{S(0)}\sec\tfrac{1}{2}(\alpha+\theta_0-\alpha_B)\sec\tfrac{1}{2}(\beta+\theta-\alpha_B)\,\mathrm{e}^{-\mathrm{i}k(r_0\cos\alpha+r\cos\beta)}\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{136}$$

The double integral in (136) could be reduced to a single integral by the method of § 7·1, but the factor preceding it is so small in practice (vanishing for θ_0 or θ equal to π) that the whole expression may be neglected. In other words, for positions of the receiver between the transmitter and the boundary, only the geometrical optics term contributes effectively to the field, which is therefore virtually the same as that pertaining to a homogeneous earth.*

7·4. *Continuity of the field across* $\theta+\theta_0=\pi$

No attempt will be made to get numerical results for arbitrary elevations of the transmitter and receiver directly from the formulae given above, and in the next section we proceed to a discussion of the ground-to-ground field. It is, however, desirable to check that, in the general case, the solution is continuous across $\theta+\theta_0=\pi$, particularly in view of the fact that the subsequent analysis centres on an examination of the field at points on this line.

The discontinuity in the geometrical optics term (99), found by subtracting its value at $\theta+\theta_0=\pi-\epsilon$ from that at $\theta+\theta_0=\pi+\epsilon$, where $\epsilon\to 0$, is

$$H_z^g\Big]_{\pi-}^{\pi+}=\Delta(S,\theta). \tag{137}$$

It is not difficult to show that (137) is balanced by the discontinuity in the diffraction term. This latter arises solely from the first expression on the right-hand side of (133); in fact, using (118) and noting that $B=0$ on $\theta+\theta_0=\pi$,

$$I\Big]_{\pi-}^{\pi+}=2\sqrt{2}\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\cot\tfrac{1}{2}(\theta_0-\alpha_B)\int_0^\infty\frac{\mathrm{e}^{-kR_1\lambda^2}}{\lambda^2+2\mathrm{i}\cot^2\frac{1}{2}(\theta_0-\alpha_B)}\,\mathrm{d}\lambda. \tag{138}$$

* Feinberg (1946) gives a second-order correction term in this case for the ground-to-ground field

The integral in (138) can be expressed (exactly) in terms of the complex Fresnel integral (20) (Ott 1943; Clemmow 1951); we have

$$I\Big]_{\pi-}^{\pi+} = 2\sqrt{\pi}\,e^{-\frac{1}{4}i\pi} F\{\sqrt{(2kR_1)}\cot\tfrac{1}{2}(\theta_0-\alpha_B)\}, \tag{139}$$

and this can be legitimately replaced by

$$I\Big]_{\pi-}^{\pi+} = 2\sqrt{\pi}\,e^{-\frac{1}{4}i\pi}\sin\tfrac{1}{2}(\theta_0-\alpha_B)\,F\{\sqrt{(2kR_1)}\cos\tfrac{1}{2}(\theta_0-\alpha_B)\} \tag{140}$$

to the order of approximation to which we are working. The corresponding discontinuity in H_z^d is now obtained from (106), (128) and (140). It is

$$H_z^d\Big]_{\pi-}^{\pi+} = \frac{2\sqrt{2i}}{\cos\frac{1}{2}(\theta-\alpha_B)\,L_2(-\cos\theta)\,L_2(\cos\theta)}\,e^{-ikS}F\{\sqrt{(2kS)}\sin\tfrac{1}{2}(\theta+\alpha_B)\}$$
$$= -\Delta(S,\theta), \quad \text{using (102) and (21).} \tag{141}$$

We remark that the necessity here for replacing (139) by (140) only arises because, owing to the greater complexity of the analysis, our method of approximation in the mixed-path problem has been slightly less refined than that adopted in the case of a homogeneous earth.

8. Transmitter and receiver on the earth's surface

8·1. *The general expression*

The analysis of §7 is now applied to a discussion of the ground-to-ground field.

We consider first the case in which the transmitter and receiver are on opposite sides of the boundary. Then

$$\theta = 0, \quad \theta_0 = \pi, \quad R = S = R_1 = d; \tag{142}$$

and, from (118)

$$a = e^{-\frac{1}{4}i\pi}\sqrt{(2r_0/d)}\tan\tfrac{1}{2}\alpha_B, \quad B = 0. \tag{143}$$

In view of the remarks concerning the interpretation of (135), it is convenient to use the formulae appropriate to $\theta+\theta_0 = \pi-\epsilon$ $(\epsilon\to 0)$, so that $H_z^{d1} = H_z^{d2}$. The geometrical optics term is thus

$$H_z^g = \frac{2\,e^{-ikd}}{\sqrt{(kd)}}, \tag{144}$$

and the diffraction term, from (106), (128), (133) and (102), is

$$H_z^d = -4\sqrt{(2/\pi)}\,e^{-\frac{1}{4}i\pi}\tan\tfrac{1}{2}\alpha_B\,e^{-ikd}\,I, \tag{145}$$

where

$$I = -\sqrt{2}e^{-\frac{1}{4}i\pi}\tan\tfrac{1}{2}\alpha_B\int_0^\infty \frac{e^{-kd\lambda^2}}{\lambda^2+2i\tan^2\frac{1}{2}\alpha_B}\,d\lambda$$
$$+e^{-\frac{1}{4}i\pi}\sqrt{\left(\frac{2r}{d}\right)}\tan\tfrac{1}{2}\alpha_B\,e^{2ikr_0\tan^2\frac{1}{2}\alpha_B}\int_{e^{\frac{1}{4}i\pi}\sqrt{(2r_0/d)}\tan\frac{1}{2}\alpha_B}^{\infty}\frac{e^{-kd\lambda^2}}{\lambda^2+2i(r/d)\tan^2\frac{1}{2}\alpha_B}\,d\lambda. \tag{146}$$

Now the first term in I is minus a half of (138) with $\theta_0 = \pi$. From (141), therefore, the corresponding term in H_z^d is $\Delta(d,0)$. Hence the complete field is given by

$$H_z = \frac{2\,e^{-ikd}}{\sqrt{(kd)}} + \Delta(d,0) + H_z^s, \tag{147}$$

where

$$H_z^s = \frac{8\,e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}\sqrt{(kr)}\tan^2\tfrac{1}{2}\alpha_B\,e^{-ikd}\,e^{2ikr_0\tan^2\frac{1}{2}\alpha_B}\int_{\sqrt{(2kr_0)}\tan\frac{1}{2}\alpha_B}^{\infty}\frac{e^{-i\lambda^2}}{\lambda^2+2kr\tan^2\frac{1}{2}\alpha_B}\,d\lambda. \tag{148}$$

The first two terms of (147) represent the field which would exist for a homogeneous earth, that is to say, in the absence of the conducting sheet; the scattered field generated by currents induced in this sheet is therefore given by H_z^s, as the notation implies. Formulae (147) and (148) are applicable when the transmitter and receiver are on opposite sides of the boundary; if they are both on the same side of the boundary the diffraction field is to be neglected altogether (as shown in §7·3).

At this stage it is convenient to introduce the parameters

$$\gamma_0 = \surd(\tfrac{1}{2}kd)\sin\alpha_B, \tag{149}$$

$$\gamma_{0t} = \surd(\tfrac{1}{2}kr_0)\sin\alpha_B. \tag{150}$$

The former appeared previously in the analysis for a homogeneous earth (equation (23)), $-\mathrm{i}\gamma_0^2$ being the 'numerical distance' of the receiver from the transmitter; correspondingly, $-\mathrm{i}\gamma_{0t}^2$ is the 'numerical distance' of the transmitter from the boundary. As might have been anticipated, the quantities (149) and (150) turn out to be the natural ones in which to express the present results; they may replace, respectively, the essentially equivalent forms $\surd(2kd)\tan(\tfrac{1}{2}\alpha_B)$ and $\surd(2kr_0)\tan(\tfrac{1}{2}\alpha_B)$ which are explicit in (148), the slight discrepancy being due to the method of approximation.

For the sake of brevity we also write

$$K(a) = 1-2\mathrm{i}aF(a) \tag{151}$$

as in (24), and

$$G(a,b) = b\,\mathrm{e}^{\mathrm{i}a^2}\int_a^\infty \frac{\mathrm{e}^{-\mathrm{i}\lambda^2}}{\lambda^2+b^2}\mathrm{d}\lambda. \tag{152}$$

Now let A be the factor by which the free-space field must be multiplied to give the field in the presence of the earth. Then our results for the ground-to-ground field, when the transmitter is situated over medium 1, may be stated thus:

for points on the same side of the boundary as the transmitter (cf. (24))

$$A = 2K(\gamma_0); \tag{153}$$

for points on the opposite side of the boundary to the transmitter

$$A = 2K(\gamma_0) + \frac{4\,\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}}{\surd\pi}\gamma_0 G\{\gamma_{0t}, \surd(\gamma_0^2-\gamma_{0t}^2)\}. \tag{154}$$

The formulae (153) and (154) can be assumed to be independent of the nature of the (vertically polarized) transmitter. The whole analysis could certainly have been carried through for a line-source with a polar diagram other than circular, and the same results obtained. But more important is the contention that (153) and (154) are also applicable to a point-source, giving the field variation in any direction which is not too oblique to the boundary, provided all distances are measured along the appropriate radius from the source. The belief that this is so is based on an examination of the known solutions for a dipole transmitter in the simpler, allied problems of propagation over a homogeneous earth (§3) and diffraction by a perfectly conducting half-plane (Senior 1953); and is supported by Feinberg's (1946) analysis.

8·2. *A special case*

The significance of (154) is most readily appreciated by considering the conditions under which some simplification is possible.

It can be shown (Clemmow 1950*b*) that it is permissible to put λ equal to its lower limit value, namely, γ_{0t}, in the non-exponential factor of the integrand provided that $|\gamma_0| \gg 1$. In the case, then, for which the receiver is a large 'numerical distance' from the transmitter (relative to medium 1), (154) becomes

$$A = -\frac{2i}{kd\sin^2\alpha_B} + \frac{4\,e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}\sqrt{\left(\frac{r}{d}\right)}F(\gamma_{0t}). \tag{155}$$

From an inspection of (155) we can follow qualitatively what happens as the receiver starts at the boundary and proceeds away from the transmitter and off to infinity over the perfectly conducting sheet. When $\sqrt{(r/d)}$ is sufficiently small, the first term, which represents the field in the absence of the conducting sheet, predominates; in fact, (153) and (154) give a smooth transition across $r = 0$, although the asymptotic approximations on which §7 is based can only be expected to apply at distances of greater than half a wave-length, say, from the boundary. But since the first term is itself small, the second term very soon takes over, and consequently there is a rapid increase of field-strength with distance in the region just beyond the boundary, a recovery effect. Finally, when $\sqrt{(r/d)} \doteqdot 1$, (155) becomes effectively

$$A = \frac{4\,e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}F(\gamma_{0t}); \tag{156}$$

in this last case, therefore, the field is equivalent to that of a transmitter in the presence of an *infinite* perfectly conducting sheet whose power and phase are modified in accordance with (156); a result which confirms the obvious supposition (P. P. Eckersley 1930; Millington 1949*b*) that at points sufficiently remote from the boundary the rate of attenuation must be characteristic of the relevant medium.

We are assuming $|\gamma_0| \gg 1$; hence, when kr is relatively small $|\gamma_{0t}|$ must be large, but as kr increases this is no longer necessary and so (156) is applicable for virtually all values of γ_{0t}; in particular, it may be noted that (156) reduces to $A = 2$ when $\gamma_{0t} = 0$, implying, as would be expected, that the field is unaffected by the imperfectly conducting medium when the transmitter is sufficiently close to the boundary (though, again, the results cannot be granted quantitative recognition unless $kr \gg 1$).

8·3. *A numerical example*

In illustration of the foregoing remarks we take a simple numerical example which has been considered briefly elsewhere (Clemmow 1950*a*). The most interesting effect to demonstrate is the field-strength recovery, and to emphasize this feature we choose medium 1 to be a pure dielectric with $\sin\alpha_B = \frac{1}{3}$ (corresponding to a dielectric constant of 8), although the conditions of the problem are then such as would scarcely be met in practice. If we assume that $r_0 = 300\lambda$ (where λ is the wave-length), γ_{0t}, from (150), is just greater than 10, a value certainly large enough to allow the Fresnel integral in (155) to be replaced by the first term of its asymptotic expansion. The complete field is thus given by

$$A = -\frac{2i}{kd\sin^2\alpha_B} \qquad \text{for } d < r_0, \tag{157}$$

$$A = -\frac{2i}{kd\sin^2\alpha_B} + 2\sqrt{\frac{2}{\pi}}\,e^{-\frac{1}{4}i\pi}\,\frac{\sqrt{(r/r_0)}}{\sqrt{(kd)}\sin\alpha_B} \qquad \text{for } d > r_0, \tag{158}$$

with the reservation that these expressions are not applicable for points inside a region about a wave-length in width centred on $d = r_0$.

Figures 10 and 11 show, respectively, plots of field-strength and phase against d/λ, using (157) and (158) in conjunction with the values of $\sin\alpha_B$ and r_0 given above. The graphs are appropriate to a point-source, and the corresponding curves relating to a homogeneous pure dielectric earth with $\sin\alpha_B = \frac{1}{3}$ and a perfectly conducting earth are also given.

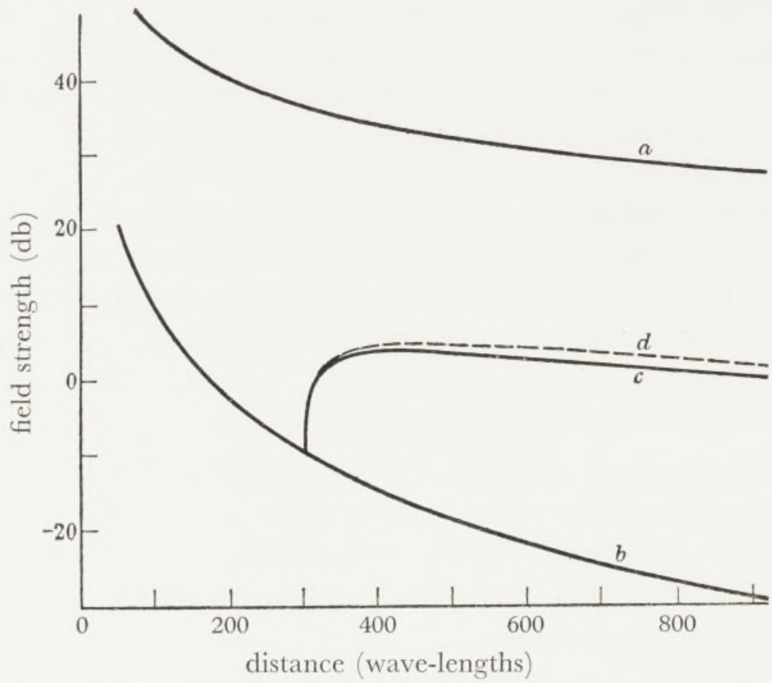

FIGURE 10. Field-strength (in decibels above an arbitrary level) against distance (in wave-lengths) from the transmitter (*a*) for a homogeneous, perfectly conducting earth (*b*) for a homogeneous, pure dielectric earth ($\sin\alpha_B = \frac{1}{3}$), (*c*) for pure dielectric earth ($\sin\alpha_B = \frac{1}{3}$) up to 300 wave-lengths from the transmitter and perfectly conducting earth beyond, by the present method, (*d*) for the conditions as in (*c*), by Millington's method.

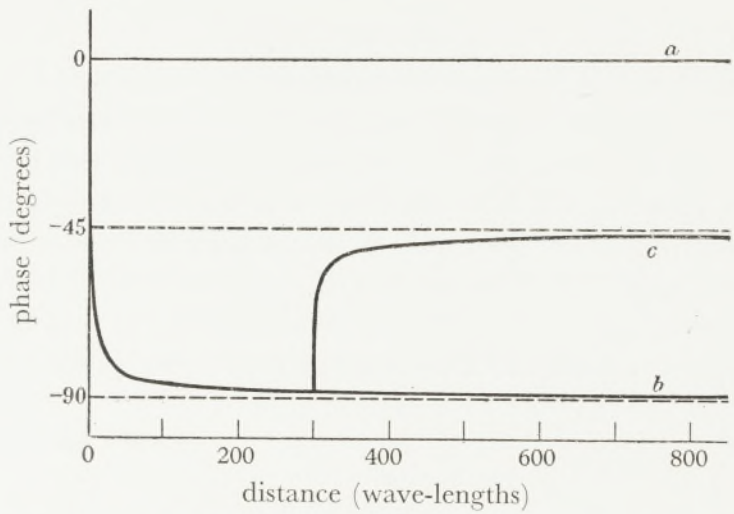

FIGURE 11. Phase (in degrees, relative to that of the free-space field) against distance (in wave-lengths) from the transmitter (*a*) for a homogeneous, perfectly conducting earth, (*b*) for a homogeneous, pure dielectric earth ($\sin\alpha_B = \frac{1}{3}$), (*c*) for pure dielectric earth ($\sin\alpha_B = \frac{1}{3}$) up to 300 wave-lengths from the transmitter and perfectly conducting earth beyond.

Referring to figure 10, we see that the initial recovery of field-strength is extremely rapid; when $r = \lambda$ it is 3·4 db above the value at the boundary, and this figure rises to a local maximum of 13·7 db when $r = 130\lambda$. The mixed-path curve obtained by Millington's method (shown dashed in the figure) lies remarkably close to that given by (158); the details of this agreement are examined in §10.

The phase plotted in figure 11 is that relative to the phase of the transmitter in free-space; here again there is a rapid climb just beyond the boundary, which means that the phase velocity in this region greatly exceeds that of free-space propagation; it is also interesting to note that the final asymptotic value of the curve, $-\frac{1}{4}\pi$, lies half-way between that appropriate to a perfectly conducting earth (namely, 0) and that appropriate to a homogeneous pure dielectric earth (namely, $-\frac{1}{2}\pi$).

9. Elevated transmitter and receiver

9·1. *The application of ray theory*

In the case of a homogeneous earth a particularly simple result with an obvious physical appeal is that of ray theory, given by (26). From the nature of this formula one might at first sight be tempted to infer, with respect to the mixed-path problem, that geometrical optics would be adequate in those regions governed by ray theory not in the immediate vicinity of OI (figure 4); but this is by no means entirely the case in the sense in which we have used these terms. Referring to the inequality (27) we note the perhaps rather surprising fact that the validity of (26) depends only on the combined heights of the transmitter and receiver, and *not* on the angle of elevation ψ; in consequence, ray theory may easily have practical application in ground-wave communication, and furthermore, as stressed in §3, may sometimes be linked with the height-gain function. On the other hand, it is well known that geometrical optics can only give a reasonable approximation at large *angles* of diffraction, and generally speaking these fall outside the limits of interest in the propagation problem.

For a perfectly conducting earth, the exact result is given by ray theory with a reflexion coefficient of +1. It is therefore reasonable to suppose that there will be some approximation in the mixed-path analysis which is valid when the inequality (27) is satisfied, although, from what has just been said, the complications associated with diffraction must still be expected to remain. The approximation is not hard to find; it consists of putting $\alpha = \theta_0$, $\beta = \theta$ in the factor $\{L_1(\cos\alpha)\, L_1(\cos\beta)\}^{-1}$ in the integrand of (98). This procedure is mathematically analogous to putting $\alpha = 0$ in the function $\rho'\{\sin(\psi-\alpha)\}$ in the integrand of (17), which was seen to lead to (26) in the case of a homogeneous earth; it defines our use of the term 'ray theory' in the present context.

Making the above-mentioned approximation, the diffraction field becomes

$$H_z^d = \frac{e^{\frac{1}{4}i\pi}}{\pi\sqrt{(2\pi)}}\frac{1}{L_1(\cos\theta_0)\, L_1(\cos\theta)}\int_{S(0)}\int_{S(0)}\frac{\cos\frac{1}{2}(\alpha+\theta_0)\cos\frac{1}{2}(\beta+\theta)}{\cos(\alpha+\theta_0)+\cos(\beta+\theta)}\, e^{-ik(r_0\cos\alpha + r\cos\beta)}\, d\alpha\, d\beta. \quad (159)$$

If medium 1 were free-space ($n = 1$), L_1 would be unity, from (80), and the solution of the problem is known. Referring to some results given elsewhere (Clemmow 1950*c*), it can therefore be deduced immediately that

$$H_z^d = -\frac{\sqrt{(2/\pi)}\, e^{\frac{1}{4}i\pi}}{L_1(\cos\theta_0)\, L_1(\cos\theta)}\left\{\frac{F[\sqrt{\{k(R_1-R)\}}]}{\sqrt{\{k(R_1+R)\}}} \pm \frac{F[\sqrt{\{k(R_1-S)\}}]}{\sqrt{\{k(R_1+S)\}}}\right\} e^{-ikR_1}, \quad (160)$$

with the upper sign for $\theta+\theta_0<\pi$ and the lower sign for $\theta+\theta_0>\pi$. The corresponding geometrical optics term is, of course,

$$H_z^g=\begin{cases}\dfrac{e^{-ikR}}{\surd(kR)}+\dfrac{e^{-ikS}}{\surd(kS)} & \text{for } \theta+\theta_0<\pi,\\[2ex] \dfrac{e^{-ikR}}{\surd(kR)}+\rho(\sin\psi)\dfrac{e^{-ikS}}{\surd(kS)} & \text{for } \theta+\theta_0>\pi.\end{cases} \tag{161}$$

Again, it is easy to check that the total field is continuous across $\theta+\theta_0=\pi$; for on this line (160) clearly reduces to

$$H_z^d=\{1-\rho(\sin\theta)\}\left\{-\sqrt{\frac{2}{\pi}}\,e^{\frac{1}{4}i\pi}\frac{F[\surd\{k(S-R)\}]}{\surd\{k(S+R)\}}\mp\frac{1}{2\surd(kS)}\right\}e^{-ikS}, \tag{162}$$

with the upper sign for $\theta+\theta_0=\pi-\epsilon$ and the lower sign for $\theta+\theta_0=\pi+\epsilon$, where $\epsilon\to 0$; and the discontinuities in (161) and (162) are indeed seen to counterbalance one another.

Formula (160) emphasizes the diffraction nature of the problem. The factor contained in the curly bracket is readily computed, being expressed in terms of Fresnel integrals whose arguments are real, but the necessity for evaluating L_1 still presents a stumbling block, as in the general case. We have seen that the difficulty is avoided if we accept the condition $\theta+\theta_0=\pi$, and by this means we shall be able to link the field of ray theory with the ground-to-ground field via the height-gain function. But before proceeding to a discussion on these lines in §9·2, a further possible simplification is worth mentioning.

It was pointed out in §4 that ray theory can sometimes be used in conjunction with an effective reflexion coefficient of -1; that is to say, the reflexion coefficient is virtually independent of the angle of incidence over the range of angles involved. This suggests that, under suitable conditions, the factor $\{L_1(\cos\theta_0)\,L_1(\cos\theta)\}^{-1}$ in (160) might be assumed independent of θ_0 and θ and given the value 2, as indicated by (80). It is interesting to note that this would lead to precisely the same result as the rigorous solution to the problem of two line-sources at T and T' in the presence of the perfectly conducting sheet but in otherwise free-space, the source at T being associated with the primary wave (3), and that at T' with the primary wave

$$H_z=-\sqrt{\frac{\pi}{2}}\,e^{-\frac{1}{4}i\pi}\,H_0^{(2)}(kS). \tag{163}$$

It is, indeed, reasonable to suppose that, in the particular circumstances now assumed, this model will furnish a good approximation to the solution of the mixed-path problem in the appropriate region above $y=0$; for it gives a continuous field which is closely that pertaining to a homogeneous earth of medium 1 for points on the same side of the diffracting edge as the transmitter, and which satisfies the boundary condition on the perfectly conducting sheet. An extension to this point of view is mentioned in §18.

9·2. *Height-gain considerations*

It is natural to suppose that the use of a height-gain function is valid under certain conditions in the mixed-path problem. A numerical example now to be given shows that this is indeed the case, the procedure being to link the ground-to-ground field with that of ray theory. An overall check on the analysis is thus obtained which is particularly reassuring in view of the fact that no simple mathematical relation between (147) and (160), (161) appears on the surface.

To make the calculation feasible we must maintain the condition $\theta+\theta_0=\pi$; this is no real restriction, for it simply means that height variations must apply to both the transmitter and the receiver, which is in any case necessary in order to introduce the height-gain function in a form explicitly related to the two media in question. Our object is most conveniently achieved by taking $x=x_0$, $y=y_0=h$.

Let us first apply the idea to a homogeneous earth. As in the example of § 8·3, we consider a pure dielectric earth with $\sin\alpha_B=\frac{1}{3}$ and take $d=600\lambda$. The inequalities (27) and (32) indicate that both ray theory and the height-gain function should be applicable for $h=2\lambda$.

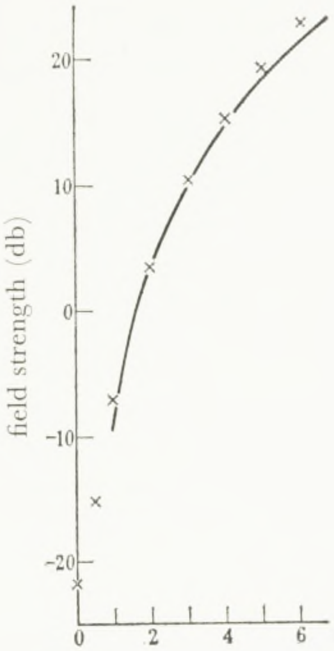

Figure 12. Field-strength (in decibels above an arbitrary level) against the common height (in wave-lengths) of transmitter and receiver situated 600 wave-lengths apart over a homogeneous, pure dielectric earth ($\sin\alpha_B=\frac{1}{3}$). Full-line curve deduced from ray theory; crosses deduced from the height-gain function, using the field-strength given by ray theory at the height of 2 wave-lengths as a starting point.

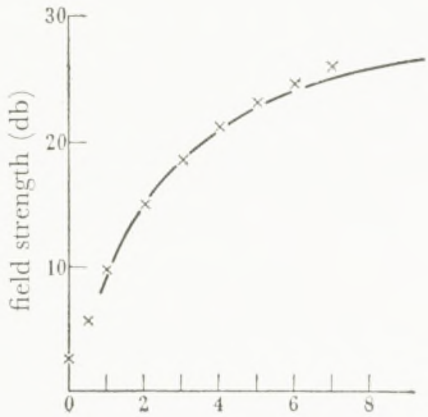

Figure 13. Field-strength (in decibels above an arbitrary level) against the common height (in wave-lengths) of transmitter and receiver, situated respectively over a pure dielectric earth ($\sin\alpha_B=\frac{1}{3}$) and a perfectly conducting earth, each being 300 wave-lengths from the boundary. Full-line curve deduced from ray theory; crosses deduced from the height-gain function, using the field-strength given by ray theory at the height of 2 wave-lengths as a starting point.

The situation is depicted in figure 12, where the field-strength is plotted against h/λ; the full-line curve is that given by formula (26), and the crosses represent points deduced from (34) starting from that on the full-line curve corresponding to $h=2\lambda$. The field-strength at $h=0$ derived in this way is in agreement with that given by (25).

Similar results for the mixed-path problem are shown in figure 13. The height-gain factor is now

$$1+\mathrm{i}kh\sin\alpha_B, \tag{164}$$

the contribution from the perfect conductor being unity. The full-line curve is computed from the ray-theory formula for the case $\theta+\theta_0=\pi$, which, from (161) and (162), may be written

$$A = 1+\left\{\tfrac{1}{2}[1+\rho(\sin\theta)]-\frac{e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}[1-\rho(\sin\theta)]\,F[\sqrt{\{k(S-R)\}}]\right\}e^{-ik(S-R)}. \tag{165}$$

The crosses represent points deduced from (164), starting from that on the full-line curve corresponding to $h=2\lambda$. Good agreement is obtained over the expected range of heights, and the field-strength at $h=0$ derived in this way agrees with that obtained directly from (158) to within 0·1 db.

10. A comparison with Millington's method

In the example of § 8·3 it appeared that the ground-to-ground field-strength curve given by Millington's method lay remarkably close to that obtained from the analysis of the present paper. This is rather surprising in view of the fact that Millington's procedure has no *ab initio* theoretical justification in the case under consideration, and it is therefore of interest to examine the reasons for its success in more detail. For the purpose of comparison, it is convenient to express the idea behind the graphical manipulation of the attenuation curves (Millington 1949*b*) analytically in terms of the complete ground-to-ground field; the formal extension is immediate, and, applied to two media (the transmitter and receiver being on opposite sides of the boundary), gives the field

$$H_z = \sqrt{\left\{\frac{H_{1z}(r_0)\,H_{2z}(r)}{H_{2z}(r_0)\,H_{1z}(r)}\,H_{1z}(d)\,H_{2z}(d)\right\}}, \tag{166}$$

where H_{1z} and H_{2z} refer to homogeneous earths of media 1 and 2 respectively. When medium 2 has infinite conductivity, (166) is equivalent to

$$A = \sqrt{\{2A_1(r_0)\,A_1(d)/A_1(r)\}}, \tag{167}$$

where A is the factor by which the free-space field must be multiplied to give the actual field, A_1 referring to a homogeneous earth of medium 1. We may remind ourselves that $A_1(0)=2$.

To facilitate the comparison between (167) and (154), we suppose that $|\gamma_0|\gg 1$, as in § 8·2. Using (24) and (25), formula (167) then reads

$$A = 2\sqrt{\left\{-\frac{2iK(\gamma_{0t})}{kd\sin^2\alpha_B\,A_1(r)}\right\}}. \tag{168}$$

We consider two limiting cases:

(*a*) r represents a small 'numerical distance' relative to medium 1, so that $|\gamma_{0t}|\gg 1$ and $\sqrt{(r/r_0)}\ll 1$. A little reduction shows that (168) is now approximately

$$-\frac{i}{\gamma_{0t}^2}\left(1+\tfrac{1}{2}\sqrt{\pi}\,e^{-\frac{1}{4}i\pi}\gamma_{0t}\sqrt{\frac{r}{r_0}}\right), \tag{169}$$

whereas the analogous expression from (158) is

$$-\frac{i}{\gamma_{0t}^2}\left(1+\frac{2\,e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}\gamma_{0t}\sqrt{\frac{r}{r_0}}\right). \tag{170}$$

(*b*) r represents a large 'numerical distance' relative to medium 1. Then (168) simplifies to

$$2\sqrt{\left\{\frac{r}{d}K(\gamma_{0t})\right\}}, \tag{171}$$

and (155) to

$$\frac{4\,e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}\sqrt{\left(\frac{r}{d}\right)}F(\gamma_{0t}). \tag{172}$$

It is clear from these formulae that the success of Millington's method in the present instance arises from the approximate numerical equality of certain functions which are mathematically quite distinct. For (169) initiates a recovery of field-strength just beyond the boundary only slightly less violent in degree than that determined by (170). Likewise, the relations

$$K^{\frac{1}{2}}(\gamma_{0t}) \doteqdot 1-\tfrac{1}{2}\sqrt{\pi}\,e^{\frac{1}{4}i\pi}\gamma_{0t} \quad \text{for } |\gamma_{0t}|\ll 1, \tag{173}$$

$$\sim e^{-\frac{1}{4}i\pi}/(\sqrt{2}\,\gamma_{0t}) \quad \text{for } |\gamma_{0t}|\gg 1, \tag{174}$$

and

$$\frac{2\,e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}F(\gamma_{0t}) \doteqdot 1-\frac{2}{\sqrt{\pi}}e^{\frac{1}{4}i\pi}\gamma_{0t} \quad \text{for } |\gamma_{0t}|\ll 1, \tag{175}$$

$$\sim e^{-\frac{1}{4}i\pi}/(\sqrt{\pi}\gamma_{0t}) \quad \text{for } |\gamma_{0t}|\gg 1 \tag{176}$$

indicate that (171) and (172) are in close agreement for all values of γ_{0t}; in particular, (174) and (176) differ only by a factor $\sqrt{(\frac{1}{2}\pi)}$, and consequently, in the example of § 8·3, Millington's curve lies merely about 2 db above our own for all points beyond a certain distance from the boundary, as figure 10 shows.

Since Millington's method receives its severest test (for an earth of two media) when applied to the model of figure 3, we expect that it will prove even more efficacious in the case of two finitely conducting media, a contention which is borne out in part II of this paper; and furthermore, our confidence is strengthened in the likelihood of it producing satisfactory results in more general problems, involving several different media and the curvature of the earth's surface, to which it is so readily adaptable.

PART II. TWO ARBITRARY MEDIA

11. The idealized problem: approximate boundary conditions

In this second part of the paper we treat a generalization of the mixed-path problem already considered, in that the assumption of perfect conductivity for one of the media is waived. This extends the range of application of the theory, and enables a comparison to be made with the one controlled experiment carried out at sufficiently short distances for the earth to be considered flat.

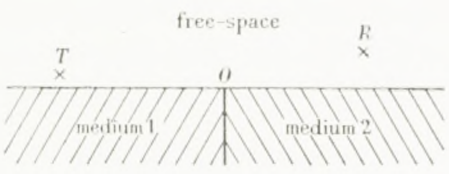

Figure 14. A possible model.

The idealized (two-dimensional) model that might be chosen is illustrated in figure 14. With the co-ordinate system as before (figure 5), the earth occupies the region $y<0$, and now consists of two homogeneous parts, medium 1 in $x<0$ and medium 2 in $x>0$. However, the introduction in this configuration, or any like it, of a second surface of discontinuity (that between medium 1 and medium 2) appears to put the exact solution of the problem beyond the reach of any mathematical technique as yet available; indeed, diffraction by a finitely conducting wedge has not so far been treated rigorously. We therefore turn to a formulation in terms of an approximate boundary condition which is likely to be valid for media, the modulus of whose (complex) dielectric constant is large, this criterion being adequately met by most types of ground. The boundary condition has been extensively used in recent years; for example, the work of Grünberg & Feinberg, previously mentioned, is based on its adoption. For our present purpose it may be stated in the following form

$$E_x = Z \sin \alpha_B H_z \quad \text{at } y = 0, \tag{177}$$

where α_B is the Brewster angle of the ground at the point in question, and $|\sin \alpha_B| \ll 1$. For 'glancing incidence' (177) is slightly more accurate than the standard form $E_x = Z \tan \alpha_B H_z$.

We give an exact solution of the problem illustrated in figure 14 in conjunction with the boundary condition (177). That is to say, at $y = 0$

$$E_x = Z \sin \alpha_{B1} H_z \quad \text{for } x < 0, \tag{178}$$

$$E_x = Z \sin \alpha_{B2} H_z \quad \text{for } x > 0, \tag{179}$$

where α_{B1}, α_{B2} are the Brewster angles of medium 1 and medium 2 respectively. The problem becomes tractable in this form because only the field in $y \geqslant 0$ is involved, and the interface between the two earth media plays no part. The solution is effected by precisely the same type of analysis as that used in part I.

An implication of (177) is that the field in $y < 0$ in the vicinity of the point in question is that of a plane wave travelling vertically downwards. The condition (177) may therefore be expected to be accurate except in some region close to the line of discontinuity at O. It might be hoped that this in turn would imply the accuracy of the corresponding solution at all points further than a fraction of a wave-length from O; and since it is only at such points that the solution can be reduced to a workable form, the limitation would be relatively unimportant. On the other hand, the degree of inaccuracy involved cannot be assessed quantitatively, and it is therefore reassuring to find that when $\sin \alpha_{B2} = 0$ (medium 2 a perfect conductor) the results are essentially in agreement with those obtained by the more rigorous treatment of part I.

In concluding this introductory section it is worth noting the slight changes that are introduced in the familiar parts of the succeeding analysis by virtue of adopting boundary conditions of the type (177). These are made clear by seeing how the analysis for a *homogeneous* earth is affected. It is apparent that the alteration in the treatment of §3 is the replacement of the exact expression (6) for the reflexion coefficient by the approximate form

$$\rho(\sin \alpha) = \frac{\sin \alpha - \sin \alpha_B}{\sin \alpha + \sin \alpha_B}. \tag{180}$$

Thus, no branch-points appear in the integrand corresponding to (9), but this is not significant since the resulting branch-cut integral was in any case neglected in arriving at (12). The further approximations that are made between (12) and the final result (22) are such that there is no essential distinction between the use of (180) rather than (6); indeed, we remark that the precise condition (177) is implicit in the second height-gain factor of (31).

The pattern now follows closely that of part I. In §12 an incident plane wave is considered, the formulation given in terms of dual integral equations, and the solution of these obtained. The deduction of the solution for a line-source follows (§13), and then the reduction of the double integrals to single integrals of the type $G(a, b)$ (§14). In §15 the general expression for the ground-to-ground field is set out and its properties noted in some special cases; a numerical example shows good agreement with an experimental result demonstrating the recovery effect (§16). Finally, the different approximate form of the solution corresponding to ray theory, valid when the transmitter and receiver are sufficiently elevated, is given in §17.

12. The solution for an incident plane wave

This section is devoted to the problem in which the plane wave (4) is incident on the interface depicted in figure 14, using the boundary conditions (178), (179). As already mentioned, only the region $y \geqslant 0$ is involved.

12·1. *The formulation in terms of dual integral equations*

The field of the incident plane wave is

$$\begin{cases} \mathbf{H}^i = (0, 0, 1)\, \mathrm{e}^{ikr\cos(\theta-\alpha)}, & (181) \\ \mathbf{E}^i = Z(\sin\alpha, -\cos\alpha, 0)\, \mathrm{e}^{ikr\cos(\theta-\alpha)}. & (182) \end{cases}$$

In order that the analysis may be paralleled with that of §5·1, the scattered field is taken as that which, to give the complete field, has to be added to that appropriate to a homogeneous earth of medium 1. If the earth were homogeneous of medium 1 there would be a reflected wave

$$\begin{cases} \mathbf{H}^r = \rho_1(\sin\alpha)\,(0, 0, 1)\, \mathrm{e}^{ikr\cos(\theta+\alpha)}, & (183) \\ \mathbf{E}^r = Z\rho_1(\sin\alpha)\,(-\sin\alpha, -\cos\alpha, 0)\, \mathrm{e}^{ikr\cos(\theta+\alpha)}, & (184) \end{cases}$$

where $\rho_1(\sin\alpha)$ is the reflexion coefficient of medium 1. As indicated in §11, in order to keep the approximations consistent the inexact form

$$\rho_1(\sin\alpha) = \frac{\sin\alpha - \sin\alpha_{B1}}{\sin\alpha + \sin\alpha_{B1}} \tag{185}$$

must be used; it will be seen shortly that this is necessary if the strictly reciprocal form of the answer is to be preserved. The scattered field is written as an angular spectrum of plane waves

$$\begin{cases} H_z^s = \int_C P(\cos\beta)\, \mathrm{e}^{-ikr\cos(\theta-\beta)}\, \mathrm{d}\beta, & (186) \\ E_x^s = -Z\int_C \sin\beta P(\cos\beta)\, \mathrm{e}^{-ikr\cos(\theta-\beta)}\, \mathrm{d}\beta, & (187) \\ E_y^s = Z\int_C \cos\beta P(\cos\beta)\, \mathrm{e}^{-ikr\cos(\theta-\beta)}\, \mathrm{d}\beta. & (188) \end{cases}$$

The total field is given by $$H_z = H_z^i + H_z^r + H_z^s. \tag{189}$$

The boundary conditions to be satisfied at $y = 0$ are (178) and (179). Expressed in terms of the scattered field these are

(I) $E_x^s = Z\sin\alpha_{B1} H_z^s$ at $y = 0$, $x < 0$;

(II) $E_x^s = Z\{\sin\alpha_{B2} H_z^s + (\sin\alpha_{B2} - \sin\alpha_{B1})\,(H_z^i + H_z^r)\}$ at $y = 0$, $x > 0$.

Using (186) and (187), (I) and (II) lead respectively to the dual integral equations

$$\int_{-\infty}^{\infty}\left\{1+\frac{\sin\alpha_{B1}}{\surd(1-\lambda^2)}\right\}P(\lambda)\,\mathrm{e}^{-\mathrm{i}kx\lambda}\,\mathrm{d}\lambda=0 \quad \text{for } x<0, \tag{190}$$

$$\int_{-\infty}^{\infty}\left\{1+\frac{\sin\alpha_{B2}}{\surd(1-\lambda^2)}\right\}P(\lambda)\,\mathrm{e}^{-\mathrm{i}kx\lambda}\,\mathrm{d}\lambda=\frac{2\surd(1-\lambda_0^2)\,(\sin\alpha_{B1}-\sin\alpha_{B2})}{\surd(1-\lambda_0^2)+\sin\alpha_{B1}}\,\mathrm{e}^{\mathrm{i}kx\lambda_0} \quad \text{for } x>0, \tag{191}$$

where $\lambda_0=\cos\alpha$. Equations (190), (191) should be compared with (63), (64); when $\sin\alpha_{B2}=0$, the only difference is that the expressions $\surd(1-\lambda^2/n^2)/n$ in the integrand of (63) and $\surd(1-\lambda_0^2/n^2)/n$ on the right-hand side of (64) are replaced by $\sin\alpha_{B1}$.

12·2. *The solution*

Using the notation and technique of §5·3, equation (190) is satisfied if

$$\left\{1+\frac{\sin\alpha_{B1}}{\surd(1-\lambda^2)}\right\}P(\lambda)=U(\lambda), \tag{192}$$

and equation (191) if

$$\left\{1+\frac{\sin\alpha_{B2}}{\surd(1-\lambda^2)}\right\}P(\lambda)=\frac{\mathrm{i}}{2\pi}\,\frac{2\surd(1-\lambda_0^2)\,(\sin\alpha_{B1}-\sin\alpha_{B2})}{\surd(1-\lambda_0^2)+\sin\alpha_{B1}}\,\frac{L(\lambda)}{L(-\lambda_0)\,(\lambda+\lambda_0)}, \tag{193}$$

where the path of integration is assumed indented above the pole at $\lambda=-\lambda_0$. To obtain $P(\lambda)$ explicitly from (192) and (193), clearly the major step is to express

$$\frac{\surd(1-\lambda^2)+\sin\alpha_{B2}}{\surd(1-\lambda^2)+\sin\alpha_{B1}} \tag{194}$$

as the product of a U-function and an L-function. We write

$$\frac{2\sin\alpha_{B1}}{\surd(1-\lambda^2)}\,\frac{\surd(1-\lambda^2)+\sin\alpha_{B2}}{\surd(1-\lambda^2)+\sin\alpha_{B1}}=\frac{1}{U_1(\lambda)\,L_1(\lambda)}, \tag{195}$$

the particular form being so chosen to reduce to (80) when $\sin\alpha_{B2}=0$, provided that in (80) $\surd(1-\lambda^2/n^2)/n$ is replaced by $\sin\alpha_{B1}$: again we note that $U_1(\lambda)=L_1(-\lambda)$. Then

$$P(\lambda)=\frac{\mathrm{i}}{2\pi}\,\frac{1-\sin\alpha_{B2}/\sin\alpha_{B1}}{\{1+\sin\alpha_{B2}(1-\lambda_0^2)^{-\frac{1}{2}}\}\{1+\sin\alpha_{B2}(1-\lambda^2)^{-\frac{1}{2}}\}}\,\frac{\surd(1+\lambda_0)\,\surd(1+\lambda)}{L_1(\lambda_0)\,L_1(\lambda)\,(\lambda+\lambda_0)}, \tag{196}$$

the symmetry in λ and λ_0 ensuring that the reciprocity condition is satisfied.

The complete field is therefore given by

$$H_z=\mathrm{e}^{\mathrm{i}kr\cos(\theta-\alpha)}+\rho_1(\sin\alpha)\,\mathrm{e}^{\mathrm{i}kr\cos(\theta+\alpha)}+H_z^s, \tag{197}$$

where

$$H_z^s=\frac{\mathrm{i}}{\pi}\left(1-\frac{\sin\alpha_{B2}}{\sin\alpha_{B1}}\right)\frac{\cos\frac{1}{2}\alpha}{L_1(\cos\alpha)\left(1+\dfrac{\sin\alpha_{B2}}{\sin\alpha}\right)}\int_C\frac{\cos\frac{1}{2}\beta\,\mathrm{e}^{-\mathrm{i}kr\cos(\theta-\beta)}}{\left(1+\dfrac{\sin\alpha_{B2}}{\sin\beta}\right)L_1(\cos\beta)\,(\cos\beta+\cos\alpha)}\,\mathrm{d}\beta. \tag{198}$$

12·3. *A transformation of the solution*

To separate the expression (197) for H_z into the sum of a geometrical optics term and a diffraction term, the path of integration C in (198) must be distorted into that of steepest descents, $S(\theta)$. The pole at $\beta=\pi-\alpha$, if captured in this process (in the positive sense), would contribute the term

$$\frac{2(\sin\alpha_{B1}-\sin\alpha_{B2})\sin\alpha\,\mathrm{e}^{\mathrm{i}kr\cos(\theta+\alpha)}}{(\sin\alpha+\sin\alpha_{B1})\,(\sin\alpha+\sin\alpha_{B2})}=\{\rho_2(\sin\alpha)-\rho_1(\sin\alpha)\}\,\mathrm{e}^{\mathrm{i}kr\cos(\theta+\alpha)}. \tag{199}$$

The solution may therefore be written

$$H_z = e^{ikr\cos(\theta-\alpha)} + \begin{pmatrix}\rho_2\\ \rho_1\end{pmatrix} e^{ikr\cos(\theta+\alpha)} + H_z^d, \tag{200}$$

where

$$\begin{pmatrix}\rho_2\\ \rho_1\end{pmatrix} = \begin{cases}\rho_2(\sin\alpha) & \text{for } 0\leqslant\theta\leqslant\pi-\alpha,\\ \rho_1(\sin\alpha) & \text{for } \pi\geqslant\theta>\pi-\alpha,\end{cases} \tag{201}$$

and

$$H_z^d = \frac{i}{\pi}\left(1-\frac{\sin\alpha_{B2}}{\sin\alpha_{B1}}\right)\frac{\cos\frac{1}{2}\alpha}{L_1(\cos\alpha)}\frac{\sin\alpha}{\sin\alpha+\sin\alpha_{B2}}\int_{S(\theta)}\frac{\sin\beta\cos\frac{1}{2}\beta\, e^{-ikr\cos(\theta-\beta)}}{(\sin\beta+\sin\alpha_{B2})\,L_1(\cos\beta)\,(\cos\beta+\cos\alpha)}\,d\beta. \tag{202}$$

13. The solution for a line-source

13·1. *The general form*

We now consider a line-source situated at (r_0, θ_0), which would, in free-space, radiate the cylindrical wave (3). Following the procedure of § 6·1 it can be seen that the solution in this case is

$$H_z = H_z^g + H_z^d, \tag{203}$$

where

$$H_z^g = \begin{cases}\sqrt{\frac{\pi}{2}}\,e^{-\frac{1}{4}i\pi}\{H_0^{(2)}(kR)+H_0^{(2)}(kS)\}+\Delta_2(S,\psi) & \text{for } 0\leqslant\theta<\pi-\theta_0,\\ \sqrt{\frac{\pi}{2}}\,e^{-\frac{1}{4}i\pi}\{H_0^{(2)}(kR)+H_0^{(2)}(kS)\}+\Delta_1(S,\psi) & \text{for } \pi\geqslant\theta>\pi-\theta_0,\end{cases} \tag{204}$$

and

$$H_z^d = \frac{e^{\frac{1}{4}i\pi}}{\pi\sqrt{(2\pi)}}\left(1-\frac{\sin\alpha_{B2}}{\sin\alpha_{B1}}\right)$$
$$\times\int_{S(0)}\int_{S(0)}\frac{\cos\frac{1}{2}(\alpha+\theta_0)\cos\frac{1}{2}(\beta+\theta)\sin(\alpha+\theta_0)\sin(\beta+\theta)\,e^{-ik(r_0\cos\alpha+r\cos\beta)}}{\{\sin(\alpha+\theta_0)+\sin\alpha_{B2}\}\{\sin(\beta+\theta)+\sin\alpha_{B2}\}L_1\{\cos(\alpha+\theta_0)\}\times L_1\{\cos(\theta+\beta)\}\{\cos(\alpha+\theta_0)+\cos(\beta+\theta)\}}\,d\alpha\,d\beta; \tag{205}$$

in (204), $\Delta_1(S,\psi)$ and $\Delta_2(S,\psi)$ are terms corresponding to (12) for media 1 and 2 respectively, where the approximate form (180) of the Fresnel reflexion coefficients is used; that is

$$\Delta_i(S,\psi) = -\sqrt{\frac{2}{\pi}}\,e^{-\frac{1}{4}i\pi}\sin\alpha_{Bi}\int_{S(0)}\frac{e^{-ikS\cos\alpha}}{\sin(\alpha+\psi)+\sin\alpha_{Bi}}\,d\alpha \quad (i=1,2). \tag{206}$$

13·2. *A simplification*

As in § 6·2, we can simplify (205) by putting $\alpha=\beta=0$ in those factors of the integrand which are 'slowly varying' for small values of α and β.

To this end we note that (195) gives

$$\frac{1}{U_1(\cos\alpha)\,L_1(\cos\alpha)} = \frac{2\sin\alpha_{B1}}{\sin\alpha}\,\frac{\sin\alpha+\sin\alpha_{B2}}{\sin\alpha+\sin\alpha_{B1}}, \tag{207}$$

and by reasoning similar to that which leads from (101) to (103), (104), it is seen that (207) implies

$$\frac{1}{U_1(\cos\alpha)} = \frac{1}{F_1(\cos\alpha)}\,\frac{\sqrt{(\sin\alpha_{B1})}}{\sin\frac{1}{2}\alpha}\,\frac{\sin\frac{1}{2}(\alpha+\alpha_{B2})}{\sin\frac{1}{2}(\alpha+\alpha_{B1})}, \tag{208}$$

$$\frac{1}{L_1(\cos\alpha)} = \frac{1}{F_2(\cos\alpha)}\,\frac{\sqrt{(\sin\alpha_{B1})}}{\cos\frac{1}{2}\alpha}\,\frac{\cos\frac{1}{2}(\alpha-\alpha_{B2})}{\cos\frac{1}{2}(\alpha-\alpha_{B1})}, \tag{209}$$

where $F_1(\cos\alpha)$ and $F_2(\cos\alpha)$ have no singularities or zeros in the finite part of the complex α-plane, and

$$F_1(\cos\alpha)\,F_2(\cos\alpha) = 1. \tag{210}$$

Substituting for $L_1(\cos\alpha)$ from (209) into (205), it is legitimate to put $\alpha=\beta=0$ in the factor $\{F_2[\cos(\alpha+\theta_0)]\,F_2[\cos(\beta+\theta)]\}^{-1}$, and thus

$$H_z^d = \frac{e^{\frac{1}{4}i\pi}}{\pi\sqrt{(2\pi)}}\frac{\sin\alpha_{B1}-\sin\alpha_{B2}}{F_2(\cos\theta_0)\,F_2(\cos\theta)} \times\int_{S(0)}\int_{S(0)}\frac{\sin\frac{1}{2}(\alpha+\theta_0)\sin\frac{1}{2}(\beta+\theta)\cos\frac{1}{2}(\alpha+\theta_0)\cos\frac{1}{2}(\beta+\theta)\,e^{-ik(r_0\cos\alpha+r\cos\beta)}}{\sin\frac{1}{2}(\alpha+\theta_0+\alpha_{B2})\sin\frac{1}{2}(\beta+\theta+\alpha_{B2})\cos\frac{1}{2}(\alpha+\theta_0-\alpha_{B1})\cos\frac{1}{2}(\beta+\theta-\alpha_{B1})\times\{\cos(\alpha+\theta_0)+\cos(\beta+\theta)\}}\,d\alpha\,d\beta. \tag{211}$$

By analogy with (107), (108) we write

$$H_z^d = \frac{e^{\frac{1}{4}i\pi}}{4\pi\sqrt{(2\pi)}}\frac{\sin\alpha_{B1}-\sin\alpha_{B2}}{F_2(\cos\theta_0)\,F_2(\cos\theta)}(H_z^{d1}+H_z^{d2}), \tag{212}$$

where

$$H_z^{d1} = \int_{S(0)}\int_{S(0)}\frac{\sin\frac{1}{2}(\alpha+\theta_0)\sin\frac{1}{2}(\beta+\theta)\,e^{-ik(r_0\cos\alpha+r\cos\beta)}}{\sin\frac{1}{2}(\alpha+\theta_0+\alpha_{B2})\sin\frac{1}{2}(\beta+\theta+\alpha_{B2})\cos\frac{1}{2}(\alpha+\theta_0-\alpha_{B1})\times\cos\frac{1}{2}(\beta+\theta-\alpha_{B1})\cos\frac{1}{2}(\alpha-\beta+\theta_0-\theta)}\,d\alpha\,d\beta, \tag{213}$$

$$H_z^{d2} = \int_{S(0)}\int_{S(0)}\frac{\sin\frac{1}{2}(\alpha+\theta_0)\sin\frac{1}{2}(\beta+\theta)\,e^{-ik(r_0\cos\alpha+r\cos\beta)}}{\sin\frac{1}{2}(\alpha+\theta_0+\alpha_{B2})\sin\frac{1}{2}(\beta+\theta+\alpha_{B2})\cos\frac{1}{2}(\alpha+\theta_0-\alpha_{B1})\times\cos\frac{1}{2}(\beta+\theta-\alpha_{B1})\cos\frac{1}{2}(\alpha+\beta+\theta_0+\theta)}\,d\alpha\,d\beta. \tag{214}$$

14. The reduction of the solution

In this section we show how the double integrals (213), (214) can be reduced to single integrals of the type encountered in part I. Explicitly we discuss only H_z^{d2} for the case $\theta \doteqdot 0$, $\theta_0 \doteqdot \pi$.

We may write

$$H_z^{d2} = \frac{\sin\frac{1}{2}\theta_0}{\sin\frac{1}{2}(\theta_0+\alpha_{B2})\cos\frac{1}{2}(\theta-\alpha_{B1})} \times\int_{S(0)}\int_{S(0)}\frac{\sin\frac{1}{2}(\beta+\theta)\,e^{-ik(r_0\cos\alpha+r\cos\beta)}}{\cos\frac{1}{2}(\alpha+\theta_0-\alpha_{B1})\sin\frac{1}{2}(\beta+\theta+\alpha_{B2})\cos\frac{1}{2}(\alpha+\beta+\theta_0+\theta)}\,d\alpha\,d\beta. \tag{215}$$

Proceeding exactly as in §7·1, (215) reduces to

$$H_z^{d2} = \frac{4\sin\frac{1}{2}\theta_0\cos\frac{1}{2}\theta\sqrt{(r_0/R_1)}\,e^{-ikR_1}}{\sin\frac{1}{2}(\theta_0-\alpha_{B1})\sin\frac{1}{2}(\theta_0+\alpha_{B2})\cos\frac{1}{2}(\theta-\alpha_{B1})\cos\frac{1}{2}(\theta+\alpha_{B2})\sin\frac{1}{2}(\theta_0+\theta)}\int_0^\infty \rho J(\rho)\,e^{-kR_1\rho^2}\,d\rho, \tag{216}$$

where

$$J(\rho) = \int_0^\pi \frac{\{\rho\sin\phi+\sqrt{(2r/R_1)}\,e^{-\frac{1}{4}i\pi}\tan\frac{1}{2}\theta\}\,d\phi}{\{\rho\cos\phi-\sqrt{(2r_0/R_1)}\,e^{-\frac{1}{4}i\pi}\cot\frac{1}{2}(\theta_0-\alpha_{B1})\}\{\rho\sin\phi+\sqrt{(2r/R_1)}\,e^{-\frac{1}{4}i\pi}\tan\frac{1}{2}(\theta+\alpha_{B2})\}\times\left\{\rho[\sqrt{(r/R_1)}\cos\phi+\sqrt{(r_0/R_1)}\sin\phi]-e^{-\frac{1}{4}i\pi}\frac{\sqrt{(2rr_0)}}{R_1}\cot\frac{1}{2}(\theta_0+\theta)\right\}}. \tag{217}$$

Putting $z = \exp(\mathrm{i}\phi)$ in (217), we have

$$J(\rho) = -\frac{4\mathrm{i}}{\rho^2}\int_{\substack{\text{unit}\\\text{circle}}} \frac{(z^2+2\mathrm{i}b_0 z/\rho-1)\,z\,\mathrm{d}z}{(z^2-2az/\rho+1)\,(z^2+2\mathrm{i}bz/\rho-1)\,(Az^2-2Bz/\rho+C)}, \tag{218}$$

where

$$\left.\begin{aligned} a &= \mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\sqrt{(2r_0/R_1)}\cot\tfrac{1}{2}(\theta_0-\alpha_{B1}), & A &= \sqrt{(r/R_1)}-\mathrm{i}\sqrt{(r_0/R_1)},\\ b &= \mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\sqrt{(2r/R_1)}\tan\tfrac{1}{2}(\theta+\alpha_{B2}), & B &= \mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\frac{\sqrt{(2r_0r)}}{R_1}\cot\tfrac{1}{2}(\theta+\theta_0),\\ b_0 &= \mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\sqrt{(2r/R_1)}\tan\tfrac{1}{2}\theta, & C &= \sqrt{(r/R_1)}+\mathrm{i}\sqrt{(r_0/R_1)}. \end{aligned}\right\} \tag{219}$$

The integral (218) can be evaluated by Cauchy's residue theorem. Thus

$$\begin{aligned} J(\rho) = 2\pi\mathrm{i}\sqrt{\frac{R_1}{r_0}}\Bigg\{ &\frac{\sqrt{(\rho^2-B^2)}\pm B\sqrt{(r_0/r)}\pm b_0\sqrt{(R_1/r)}}{\sqrt{(\rho^2-B^2)}\,\{\sqrt{(\rho^2-B^2)}\pm B\sqrt{(r_0/r)}\pm b\sqrt{(R_1/r)}\}\,\{\sqrt{(\rho^2-B^2)}\pm a\sqrt{(R_1/r_0)}\mp B\sqrt{(r/r_0)}\}}\\ &-\frac{\sqrt{(\rho^2-a^2)}+b_0}{\sqrt{(\rho^2-a^2)}\,\{\sqrt{(\rho^2-a^2)}+b\}\,\{\sqrt{(\rho^2-a^2)}+a\sqrt{(r/r_0)}-B\sqrt{(R_1/r_0)}\}}\Bigg\}\\ &-2\pi\mathrm{i}\sqrt{\frac{R_1}{r}}\frac{b-b_0}{\sqrt{(\rho^2-b^2)}\,\{\sqrt{(\rho^2-b^2)}-a\}\,\{\sqrt{(\rho^2-b^2)}-b\sqrt{(r_0/r)}-B\sqrt{(R_1/r)}\}}, \end{aligned} \tag{220}$$

where the radicals are those branches with positive real parts, and the upper sign is for $\theta+\theta_0<\pi$, the lower sign for $\theta+\theta_0>\pi$.

Substituting for $J(\rho)$ from (220) into (216) it is seen that

$$H_z^{d2} = \frac{8\pi\mathrm{i}\sin\frac{1}{2}\theta_0\cos\frac{1}{2}\theta\,\mathrm{e}^{-\mathrm{i}kR_1}}{\sin\frac{1}{2}(\theta_0-\alpha_{B1})\sin\frac{1}{2}(\theta_0+\alpha_{B2})\cos\frac{1}{2}(\theta-\alpha_{B1})\cos\frac{1}{2}(\theta+\alpha_{B2})\sin\frac{1}{2}(\theta+\theta_0)}(I_1+I_2+I_3), \tag{221}$$

where

$$I_1 = \mathrm{e}^{-kR_1B^2}\int_{\pm\mathrm{i}B}^{\infty}\frac{\{\lambda\pm[B\sqrt{(r_0/r)}+b_0\sqrt{(R_1/r)}]\}\,\mathrm{e}^{-kR_1\lambda^2}}{\{\lambda\pm[B\sqrt{(r_0/r)}+b\sqrt{(R_1/r)}]\}\,\{\lambda\pm[a\sqrt{(R_1/r_0)}-B\sqrt{(r/r_0)}]\}}\,\mathrm{d}\lambda, \tag{222}$$

with the upper sign for $\theta+\theta_0<\pi$ and the lower sign for $\theta+\theta_0>\pi$,

$$I_2 = -\mathrm{e}^{-kR_1a^2}\int_{\mathrm{i}a}^{\infty}\frac{(\lambda+b_0)\,\mathrm{e}^{-kR_1\lambda^2}}{(\lambda+b)\,\{\lambda+a\sqrt{(r/r_0)}-B\sqrt{(R_1/r_0)}\}}\,\mathrm{d}\lambda \tag{223}$$

$$I_3 = -\sqrt{\frac{r_0}{r}}(b-b_0)\,\mathrm{e}^{-kR_1b^2}\int_{\mathrm{i}b}^{\infty}\frac{\mathrm{e}^{-kR_1\lambda^2}}{(\lambda-a)\,\{\lambda-b\sqrt{(r_0/r)}-B\sqrt{(R_1/r)}\}}\,\mathrm{d}\lambda. \tag{224}$$

If we put the integrands of (222), (223), (224) into partial fractions, and then treat the resulting expression for $I_1+I_2+I_3$ to the type of transformation which leads from (131) to (133), we get

$$\begin{aligned} I_1+I_2+I_3 = N\Bigg\{&\mp[a\sqrt{(R_1/r_0)}-B\sqrt{(r/r_0)}]\,\mathrm{e}^{-kR_1B^2}\int_{\pm\mathrm{i}B}^{\infty}\frac{\mathrm{e}^{-kR_1\lambda^2}\,\mathrm{d}\lambda}{\lambda^2-[a\sqrt{(R_1/r_0)}-B\sqrt{(r/r_0)}]^2}\\ &+[a\sqrt{(r/r_0)}-B\sqrt{(R_1/r_0)}]\,\mathrm{e}^{-kR_1a^2}\int_{\mathrm{i}a}^{\infty}\frac{\mathrm{e}^{-kR_1\lambda^2}\,\mathrm{d}\lambda}{\lambda^2-[a\sqrt{(r/r_0)}-B\sqrt{(R_1/r_0)}]^2}\Bigg\}\\ +(1-N)\Bigg\{&\mp[B\sqrt{(r_0/r)}+b\sqrt{(R_1/r)}]\,\mathrm{e}^{-kR_1B^2}\int_{\pm\mathrm{i}B}^{\infty}\frac{\mathrm{e}^{-kR_1\lambda^2}\,\mathrm{d}\lambda}{\lambda^2-[B\sqrt{(r_0/r)}+b\sqrt{(R_1/r)}]^2}\\ &-[b\sqrt{(r_0/r)}+B\sqrt{(R_1/r)}]\,\mathrm{e}^{-kR_1b^2}\int_{\mathrm{i}b}^{\infty}\frac{\mathrm{e}^{-kR_1\lambda^2}\,\mathrm{d}\lambda}{\lambda^2-[b\sqrt{(r_0/r)}+B\sqrt{(R_1/r)}]^2}\\ &+a\,\mathrm{e}^{-kR_1b^2}\int_{\mathrm{i}b}^{\infty}\frac{\mathrm{e}^{-kR_1\lambda^2}\,\mathrm{d}\lambda}{\lambda^2-a^2}+b\,\mathrm{e}^{-kR_1a^2}\int_{\mathrm{i}a}^{\infty}\frac{\mathrm{e}^{-kR_1\lambda^2}\,\mathrm{d}\lambda}{\lambda^2-b^2}\Bigg\}, \end{aligned} \tag{225}$$

where the upper sign is for $\theta+\theta_0<\pi$, the lower sign for $\theta+\theta_0>\pi$, and

$$N=\frac{BR_1/\surd(r_0r)-a\surd(R_1/r_0)+b_0\surd(R_1/r)}{BR_1/\surd(r_0r)-a\surd(R_1/r_0)+b\surd(R_1/r)}. \tag{226}$$

Note that when $\sin\alpha_{B2}=0$, $N=1$ and (225) reduces to (133).

The discontinuity in H_z^d across $\theta+\theta_0=\pi$, calculated from (225), does not exactly balance that of the geometrical optics term (204). The reason for this slight discrepancy is not clear, but it can be removed by using a value of N which is different from (226) though approximately equal to it when $|\alpha_{B1}|$ and $|\alpha_{B2}|$ are small. To minimize the lengthy algebra we proceed straight to the case $\theta=0$, $\theta_0=\pi$.

15. Transmitter and receiver on the earth's surface

15·1. *The general formula*

Putting $\theta=0$, $\theta_0=\pi$, (219) gives

$$\left.\begin{aligned} a\surd(R_1/r_0)&=\surd2\,e^{-\frac14 i\pi}\tan\tfrac12\alpha_{B1},\\ b\surd(R_1/r)&=\surd2\,e^{-\frac14 i\pi}\tan\tfrac12\alpha_{B2},\\ b_0=0,\quad B&=0.\end{aligned}\right\} \tag{227}$$

To meet the above-mentioned difficulty concerning continuity across $\theta+\theta_0=\pi$, we take

$$a\surd(R_1/r_0)=\frac{1}{\surd2}e^{-\frac14 i\pi}\alpha_{B1},\quad b\surd(R_1/r)=\frac{1}{\surd2}e^{-\frac14 i\pi}\alpha_{B2}, \tag{228}$$

with the result, from (226), that

$$N=\frac{\alpha_{B1}}{\alpha_{B1}-\alpha_{B2}}. \tag{229}$$

Then (221) gives effectively

$$H_z^{d2}=8\pi i\,e^{-ikd}(I_1+I_2+I_3), \tag{230}$$

where, from (225),

$$\begin{aligned} I_1+I_2+I_3=&\frac{\alpha_{B1}}{\alpha_{B1}-\alpha_{B2}}\left\{\mp a\sqrt{\frac{R_1}{r_0}}\int_0^\infty\frac{e^{-kR_1\lambda^2}}{\lambda^2-a^2(R_1/r_0)}d\lambda+a\sqrt{\frac{r}{r_0}}e^{-kR_1a^2}\int_{ia}^\infty\frac{e^{-kR_1\lambda^2}}{\lambda^2-a^2(r/r_0)}d\lambda\right\}\\ &+\frac{\alpha_{B2}}{\alpha_{B1}-\alpha_{B2}}\left\{\pm b\sqrt{\frac{R_1}{r}}\int_0^\infty\frac{e^{-kR_1\lambda^2}}{\lambda^2-b^2(R_1/r)}d\lambda+b\sqrt{\frac{r_0}{r}}e^{-kR_1b^2}\int_{ib}^\infty\frac{e^{-kR_1\lambda^2}}{\lambda^2-b^2(r_0/r)}d\lambda\right.\\ &\left.-a\,e^{-kR_1b^2}\int_{ib}^\infty\frac{e^{-kR_1\lambda^2}}{\lambda^2-a^2}d\lambda-b\,e^{-kR_1a^2}\int_{ia}^\infty\frac{e^{-kR_1\lambda^2}}{\lambda^2-b^2}d\lambda\right\}, \end{aligned} \tag{231}$$

with the upper sign for $\theta+\theta_0<\pi$ and the lower sign for $\theta+\theta_0>\pi$.

Next, we consider H_z^{d1}. There is no longer any simple relation like (135), and the foregoing analysis must be repeated. We confine ourselves to stating the result. Corresponding to (230)

$$H_z^{d2}=8\pi i\,e^{-ikd}I_0, \tag{232}$$

where I_0 is very similar to (231), the parts within the curly brackets remaining unaltered, provided the upper sign is used, and the external factors $1/(\alpha_{B1}-\alpha_{B2})$ being replaced by $1/(\alpha_{B1}+\alpha_{B2})$.

Noting that (212) gives

$$H_z^d=\frac{e^{\frac14 i\pi}}{4\pi\surd(2\pi)}(\alpha_{B1}-\alpha_{B2})(H_z^{d1}+H_z^{d2}), \tag{233}$$

we are now in a position to write down the expression for the complete ground-to-ground field. In order to cast the solution into its most compact form we introduce the appropriate 'numerical distances' via the quantities

$$\gamma_{01}^2 = \tfrac{1}{2}kr_0\alpha_{B1}^2, \quad \gamma_1^2 = \tfrac{1}{2}kd\alpha_{B1}^2,$$
$$\gamma_{02}^2 = \tfrac{1}{2}kr\alpha_{B2}^2, \quad \gamma_2^2 = \tfrac{1}{2}kd\alpha_{B2}^2, \tag{234}$$

and also make use of the relations

$$G(0,\gamma_i) = \sqrt{\pi}\,e^{\frac{1}{4}i\pi}F(\gamma_i) \quad (i=1,2) \tag{235}$$

$$G(\gamma_{01},\gamma_{02}) + G(\gamma_{02},\gamma_{01}) = 2iF(\gamma_{01})\,F(\gamma_{02}), \tag{236}$$

the first of which has already been mentioned in going from (138) to (139), the second being proved elsewhere (Clemmow & Senior 1953).

Thus, when the transmitter and receiver are on opposite sides of the boundary, the factor A by which the free-space field must be multiplied to give the actual ground-to-ground field is

$$A = \frac{2}{\alpha_{B1}+\alpha_{B2}}\Big\{\alpha_{B1}K(\gamma_1)+\alpha_{B2}K(\gamma_2)$$
$$+\frac{2\,e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}\Big[\alpha_{B1}\gamma_1 G\Big(\gamma_{01},\gamma_{01}\sqrt{\frac{r}{r_0}}\Big)+\alpha_{B2}\gamma_2 G\Big(\gamma_{02},\gamma_{02}\sqrt{\frac{r_0}{r}}\Big)-2i\sqrt{(\alpha_{B1}\alpha_{B2}\gamma_1\gamma_2)}\,F(\gamma_{01})\,F(\gamma_{02})\Big]\Big\}. \tag{237}$$

When the receiver is on the same side of the boundary as the transmitter we may use in (213) and (214) approximations analogous to those adopted in §7·3; these give $H_z^{d1}+H_z^{d2}=0$, whence the diffraction field is negligible and the total field is effectively that pertaining to a homogeneous earth. Both this result and (237) cannot be assumed to hold within half a wave-length, say, of the boundary.

15·2. *Limiting cases: the geometric mean formula*

The formula (237) is clearly reciprocal, being unaltered by the transformation $r \leftrightarrow r_0$, $\alpha_{B1} \leftrightarrow \alpha_{B2}$. It is somewhat complicated, but may be seen to have the expected behaviour in a number of limiting cases.

(1) $\alpha_{B1}=\alpha_{B2}$. Then $\gamma_1=\gamma_2$, $\gamma_{01}\sqrt{r}=\gamma_{02}\sqrt{r_0}$, so the part in square brackets vanishes and the formula $A=2K(\gamma_1)$ for a homogeneous earth is recovered.

(2) $r=0$. Then $\gamma_{02}=0$, $\gamma_{01}=\gamma_1$, $\gamma_{02}\sqrt{(r_0/r)}=\gamma_2$, and use of (235) shows that (237) reduces to $A=2K(\gamma_{01})$, which gives the field at distance r_0 from a transmitter in the presence of a homogeneous earth of medium 1.

(3) $\alpha_{B2}=0$. Then $\gamma_{02}=\gamma_2=0$ and (237) becomes

$$A = 2K(\gamma_1)+\frac{4\,e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}\gamma_1 G\Big(\gamma_{01},\gamma_{01}\sqrt{\frac{r}{r_0}}\Big); \tag{238}$$

hence (154) is recovered, with modifications arising only from the slightly different definitions of the 'numerical distances'. This result indicates the extent to which the use of approximate boundary conditions is justified.

(4) $|\gamma_{01}|, |\gamma_{02}| \gg 1$. The asymptotic approximations

$$F(a) \sim \frac{1}{2\mathrm{i}a}, \tag{239}$$

$$G(a,b) \sim \frac{b}{2\mathrm{i}a(a^2+b^2)}, \tag{240}$$

$$K(a) \sim \frac{1}{2\mathrm{i}a^2}, \tag{241}$$

for large $|a|$ show that in this case the expression in square brackets in (237) vanishes to our order of approximation, and hence

$$A \sim -\frac{\mathrm{i}}{\alpha_{B1}+\alpha_{B2}}\left(\frac{\alpha_{B1}}{\gamma_1^2}+\frac{\alpha_{B2}}{\gamma_2^2}\right)$$

$$= -\frac{\mathrm{i}}{\alpha_{B1}+\alpha_{B2}}\frac{2}{kd}\left(\frac{1}{\alpha_{B1}}+\frac{1}{\alpha_{B2}}\right) \quad \text{from (234).}$$

This may be written

$$A \sim \sqrt{\left\{\left(-\frac{\mathrm{i}}{\gamma_1^2}\right)\left(-\frac{\mathrm{i}}{\gamma_2^2}\right)\right\}}, \tag{242}$$

which is evidently the geometric-mean formula discussed in §4. The present derivation shows that it is applicable when the distances of the transmitter and receiver from the boundary represent large 'numerical distances' relative to the *respective* media on which they are situated. On this count Millington's method for the mixed-path problem is in error; for his procedure the corresponding condition is the stricter one that the distances of the transmitter and receiver from the boundary represent large 'numerical distances' relative to *both* media.

(5) $r = r_0$. This case is mentioned here because when it holds formula (237) is expressible in terms of the Fresnel integral.

For from (236) we have

$$G(\gamma_{0i}, \gamma_{0i}) = \mathrm{i}F^2(\gamma_{0i}) \quad (i = 1, 2), \tag{243}$$

giving

$$A = \frac{2}{\alpha_{B1}+\alpha_{B2}}\left\{\alpha_{B1}K(\gamma_1)+\alpha_{B2}K(\gamma_2)-\frac{2\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{\sqrt{\pi}}\sqrt{(kr)}\,[\alpha_{B1}F(\gamma_{01})-\alpha_{B2}F(\gamma_{02})]^2\right\}. \tag{244}$$

It may be mentioned that Millington's method always leads to the geometric-mean formula when $r = r_0$, as is clear from (166).

16. A numerical example: comparison with experiment

A ground-to-ground experiment has been conducted by Millington (1949*a*; see also Millington & Isted 1950) which is ideal for comparison with the flat-earth theory given in the present paper. It was on a frequency of 77·5 mc/s (a wave-length of approximately 4 m), with a transmission path partly over land (medium 1) and partly over sea water (medium 2) having a total length of about 4 km (the further section over land again being of no concern here). The conditions are closely represented by

$$\alpha_{B1} = \tfrac{1}{4}, \quad \alpha_{B2} = \tfrac{1}{30}\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}, \quad r_0 = 350\lambda, \quad r \leqslant 550\lambda. \tag{245}$$

From (245), γ_{01} and γ_1 are real and much greater than 1, but $|\gamma_2|$ is of the order of unity; in particular, $\gamma_{02}^2\,(r_0/r) = 1{\cdot}2217\mathrm{i}$. The asymptotic expansions (239), (240), (241) were therefore used to some extent in (237), but could not be applied to $F(\gamma_{02})$, $K(\gamma_2)$ and $G(\gamma_{02}, \gamma_{02}\surd(r_0/r))$. Since $\arg\gamma_2 = \arg\gamma_{02} = \frac{1}{4}\pi$, it was possible to evaluate the first two of these with the help of the tables of

$$e^{-x^2}\int_0^x e^{\lambda^2}\,d\lambda, \tag{246}$$

given by Miller & Gordon (1931) for real values of x. The last was handled by a numerical evaluation of

$$e^{-x^2}\int_0^x \frac{e^{\lambda^2}}{\lambda^2+1\cdot2217}\,d\lambda \tag{247}$$

for real values of x between 0 and 1·5.

The results of the complete computation are illustrated in figures 15 and 16.

Figure 15 shows the attenuation curve, appropriate to a point-source, for the composite path, those for the respective homogeneous earths also being included for comparison purposes. The mixed-path curve contains a region of marked recovery, the field-strength rising very steeply just beyond the boundary to a local maximum some 10 db above its value

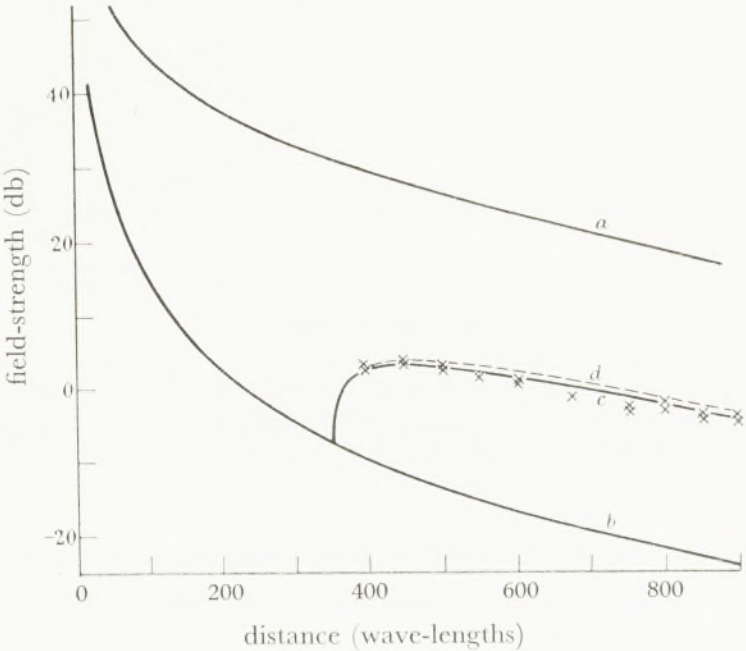

FIGURE 15. Field-strength (in decibels above an arbitrary level) against distance (in wave-lengths) from the transmitter (*a*) for a homogeneous medium, sea water with $\sin\alpha_B = \frac{1}{30}\exp(\frac{1}{4}i\pi)$, (*b*) for a homogeneous medium, land with $\sin\alpha_B = \frac{1}{4}$, (*c*) for the mixed-path, by the present method, (*d*) for the mixed-path, by Millington's method.

there, at a distance from it of about 100 wave-lengths; and is just beginning to run parallel to the 'all-sea' curve at the limit of the graph. The curve derived from Millington's procedure is shown dashed, and away from the boundary lies only about 1 db above that calculated from (237). The individual crosses are experimental points; they have been plotted relative to the dashed curve in order to allow for a slight discrepancy between the present graphs and those given by Millington, due possibly to small differences in the choice of values for α_{B1} and α_{B2}.

The corresponding phase curves are shown in figure 16. Although the phase for the 'all-sea' path is already below that for the 'all-land' path at 350 wave-lengths from the transmitter, there is nevertheless, for the composite path, a very rapid phase recovery just beyond the boundary: the curve rises steeply to a local maximum at about 20 wave-lengths from the boundary, and then settles down quickly to run parallel to the 'all-sea' curve, its ultimate asymptotic value being $-135°$.

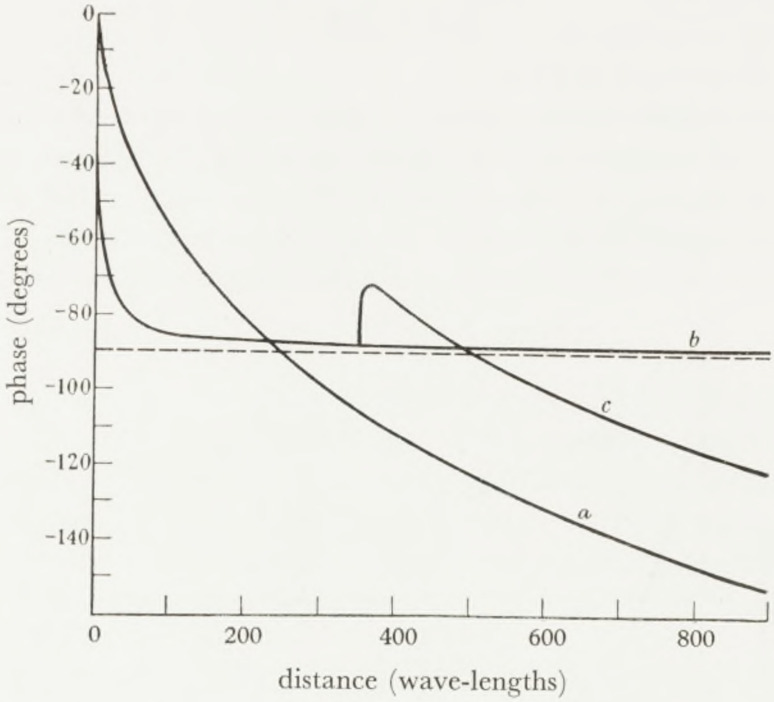

FIGURE 16. Phase (in degrees relative to that of the free-space field) against distance (in wave-lengths) from the transmitter (*a*) for a homogeneous medium, sea water with $\sin \alpha_B = \frac{1}{30} \exp (\frac{1}{4} i\pi)$, (*b*) for a homogeneous medium, land with $\sin \alpha_B = \frac{1}{4}$, (*c*) for the mixed-path.

17. Elevated transmitter and receiver: ray theory

To complete the analysis we set out briefly in this section the results of a 'ray theory' corresponding to that given in §9·1.

The appropriate steepest descents approximation to (205) for sufficiently elevated transmitter and/or receiver is (cf. (159), (160))

$$H_z^d = \frac{e^{\frac{1}{4}i\pi}}{\pi\sqrt{(2\pi)}} \frac{(\sin\alpha_{B1} - \sin\alpha_{B2}) \sin\theta_0 \sin\theta}{\sin\alpha_{B1}(\sin\theta_0 + \sin\alpha_{B2})(\sin\theta + \sin\alpha_{B2}) L_1(\cos\theta_0) L_1(\cos\theta)}$$
$$\times \int_{S(0)} \int_{S(0)} \frac{\cos\frac{1}{2}(\alpha+\theta_0)\cos\frac{1}{2}(\beta+\theta)}{\cos(\alpha+\theta_0)+\cos(\beta+\theta)} e^{-ik(r_0\cos\alpha + r\cos\beta)} d\alpha\, d\beta$$
$$= -\sqrt{\frac{2}{\pi}} e^{\frac{1}{4}i\pi} \frac{(\sin\alpha_{B1} - \sin\alpha_{B2}) \sin\theta_0 \sin\theta}{\sin\alpha_{B1}(\sin\theta_0 + \sin\alpha_{B2})(\sin\theta + \sin\alpha_{B2}) L_1(\cos\theta_0) L_1(\cos\theta)}$$
$$\times \left\{ \frac{F[\sqrt{\{k(R_1-R)\}}]}{\sqrt{\{k(R_1+R)\}}} \pm \frac{F[\sqrt{\{k(R_1-S)\}}]}{\sqrt{\{k(R_1+S)\}}} \right\} e^{-ikR_1}, \quad (248)$$

with the upper sign for $\theta+\theta_0<\pi$ and the lower sign for $\theta+\theta_0>\pi$. The corresponding geometrical optics term is

$$H_z^g=\begin{cases}\dfrac{e^{-ikR}}{\sqrt{(kR)}}+\rho_1(\sin\psi)\dfrac{e^{-ikS}}{\sqrt{(kS)}} & \text{for } \theta+\theta_0>\pi,\\ \dfrac{e^{-ikR}}{\sqrt{(kR)}}+\rho_2(\sin\psi)\dfrac{e^{-ikS}}{\sqrt{(kS)}} & \text{for } \theta+\theta_0<\pi.\end{cases} \tag{249}$$

It is easily seen that the combination of (248) and (249) is continuous across $\theta+\theta_0=\pi$. For on this line $\theta=\psi$, and from (207)

$$\frac{\sin\psi}{\sin\alpha_{B1}(\sin\psi+\sin\alpha_{B2})\,L_1(\cos\theta_0)\,L_1(\cos\theta)}=\frac{2}{\sin\psi+\sin\alpha_{B1}}, \tag{250}$$

so that (248) becomes

$$H_z^d=\{\rho_2(\sin\psi)-\rho_1(\sin\psi)\}\left\{-\sqrt{\frac{2}{\pi}}\,e^{\frac{1}{4}i\pi}\frac{F[\sqrt{\{k(S-R)\}}]}{\sqrt{\{k(S+R)\}}}\pm\frac{1}{2\sqrt{(kS)}}\right\}e^{-ikS}, \tag{251}$$

the discontinuity in which just balances that in (249). The complete field on $\theta+\theta_0=\pi$ is in fact

$$H_z=\frac{e^{-ikR}}{\sqrt{(kR)}}+\left\{\tfrac{1}{2}[\rho_1(\sin\psi)+\rho_2(\sin\psi)]+[\rho_1(\sin\psi)-\rho_2(\sin\psi)]\sqrt{\frac{2}{\pi}}\,e^{\frac{1}{4}i\pi}\frac{F[\sqrt{\{k(S-R)\}}]}{\sqrt{\{k(1+R/S)\}}}\right\}\frac{e^{-ikS}}{\sqrt{(kS)}}. \tag{252}$$

The formula (252) is a generalization of (165), the two being obviously equivalent when $\rho_2(\sin\psi)=1$.

18. Concluding remarks

The main object of this paper is to give an analytical treatment of a suitably simplified problem which is fundamental in the theory of radio propagation over an inhomogeneous earth. This purpose is achieved by establishing, with adequate rigour, formulae from which any example could be largely worked out; furthermore, these formulae are simple enough, at least in special cases, to demonstrate the general nature of the effects involved, and they provide, in particular, a theoretical confirmation of the sufficiency of Millington's method in practical application. Many other aspects, however, remain to be considered; for example, as noted in the introduction, problems of great interest arise which are allied to but somewhat different from that treated here, in addition to those involving the obvious generalizations of increasing the number of media and allowing for the curvature of the earth's surface. There are several ramifications of the present analysis which may lead to an understanding of a wider range of phenomena; it is hoped that these will be pursued in detail elsewhere, but we conclude by indicating something of their scope in a brief critique of the mathematical method.

Let us begin by considering the limitations of our method. In the first place, it applies only to a single boundary; integral equations could be set up in more general cases, but no rigorous solution then appears possible; indeed, it would seem that the separation of the surface of discontinuity, $y=0$, into two homogeneous sections extending from $x=-\infty$ to $x=0$ and from $x=0$ to $x=+\infty$, is a vital condition for the success of the exact analysis; for example, even the problem of a plane wave incident in free-space on an infinitely thin, perfectly conducting strip of finite width has not yet proved tractable to the present technique. Again, the assumption that the earth's surface is flat cannot be waived.

Turning from the question as to when a formal solution is possible, we now consider the method used for its reduction to an expression capable of yielding numerical results. Basically, the procedure is to remove certain factors from the integrand of a double integral at the 'predominant' values of the two variables of integration; exactly which factors are involved depends on the particular θ and θ_0, but part of the integrand has always to be treated in this way before any progress can be made. The extent of the error thus introduced cannot be stated with precision, but it seems that the validity of the method depends on kr and kr_0 being 'large' in the sort of way that is common in the calculation of radiation fields; this despite the fact that the resulting approximation to the solution is finite and continuous at $r = 0$ and $r_0 = 0$ $(R \neq 0)$, not having the infinity which usually indicates the failure of an asymptotic expression.* It may be remarked that the restriction is likely to be most stringent with regard to the phase, and this is especially unfortunate if it is true that coastal refraction phenomena are largely determined within a wave-length of the coastline.

The direct scope of the solution is limited by the difficulties of computation. It would, of course, be out of the question to tabulate $G(a, b)$ over the required complex range of a and b. There are, however, a number of results connecting $G(a, b)$ with the Fresnel integral (Clemmow & Senior 1953), and the prime need for facilitating the calculations is really a tabulation of this latter function, which would in any case be valuable in other problems. As a contribution to this end Clemmow & Munford (1952) have computed a four-figure table of $F[\surd(\frac{1}{2}\pi)\, a]$, $0 \leqslant |a| \leqslant 0{\cdot}8$, $0 \leqslant \arg a \leqslant 45^\circ$, at intervals suitable for linear interpolation each way; but much remains to be done to close the gap between these values of $|a|$ and those for which the asymptotic expansion is adequate.

We now discuss several means by which further results might be obtained. Perhaps the most pertinent question to ask is whether the method can be directly adapted to treat the case of a point-source. The answer is probably yes, the fundamental consideration being a suitably polarized plane wave incident at an arbitrary angle, the plane of incidence being no longer constrained to lie normal to the boundary line. At first sight it seems likely that the technique given elsewhere (Clemmow 1951, Miles 1952) for solving quasi three-dimensional diffraction problems would be applicable in this case, but a closer inspection indicates that a derivation of the complete solution meets with the following difficulty: in the problem of reflexion at the interface of two media the basic polarizations are, in the present notation, those for which $E_y = 0$ or $H_y = 0$ respectively; whereas, in diffraction problems dealing with two-dimensional conductors in free-space, whose generators are parallel to the z-axis, the basic polarizations are those for which $E_z = 0$ or $H_z = 0$ respectively; and it is not yet clear how these different aspects can be combined. On the other hand, whether or not this difficulty can be resolved, the use of approximate boundary conditions reduces the problem to a scalar one which is certainly tractable, as the work of Grünberg and Feinberg shows.

Even if the solution for a point-source were obtained it would be complicated and subject to the limitations described above, so that simpler approximate methods should certainly be considered. One approach is to apply the formulae for normal incidence to each radial

* This point might repay closer examination. For instance, the behaviour of the expression (154) at $r = 0$ is that which would be expected, from general diffraction theory, in the exact solution.

independently, as has been suggested by Millington with reference to his own work; Feinberg's analysis lends support to this idea. Again, as regards coastal refraction, the arguments of Eckersley and Ratcliffe can be more forcibly applied to phase curves such as that shown in figure 11: in physical terms, the phase velocity over land just before the boundary is negligibly less than that of free-space propagation, whereas over sea just beyond the boundary it is very much greater; Struszynski (in the Discussion following the paper by Millington & Isted (1950)) has suggested qualitatively that this will be the case by a simple argument based on the tilt of the wave-front near the earth's surface, although in the author's opinion his reasoning is not entirely unambiguous. This effect certainly implies a 'refraction' in the right direction, but the magnitude would appear to be so sensitive to the conditions of the experiment, in particular to the positions of the transmitter and receiver, that nothing further can usefully be said at this stage. Incidentally, Millington's speculation that his technique might also be applicable to phase is to some extent borne out by the analysis of this paper, though it would be liable to give errors in certain circumstances.

With reference to the 'image' method mentioned at the end of § 9·1, it might be extended by using the exact image (9) of a line-source in a homogeneous, flat earth in place of the special image (163). This procedure would avoid the introduction of the function L_1; it also offers the possibility of an approximate examination of the field in the immediate vicinity of the boundary, and is equally applicable to the case of a primary point-source without restriction on the direction of propagation. On the other hand, it is limited by the requirement that one of the media be a perfect conductor.

Finally, a word should be said about the case of horizontal polarization. The formal solution could be obtained by an analysis similar to that for vertical polarization, though its reduction to a workable form would proceed on somewhat different lines because the steepest descents technique would no longer be characterized by the existence of a pole close to the saddle-point. Alternatively, because of the invariance of Maxwell's equations under the transformation $\mathbf{E} \rightarrow \mathbf{H}$, $\mathbf{H} \rightarrow -\mathbf{E}$, $\epsilon \leftrightarrow \mu$, the general solution in part II must yield that for horizontal polarization (in terms of E_z rather than H_z) on writing $1/\alpha_{B1}$ for α_{B1} and $1/\alpha_{B2}$ for α_{B2}. For the ground-to-ground field the geometric-mean formula would be valid for all positions of the receiver on the opposite side of the boundary to the transmitter except those very close to it. On the other hand, the height-gain is so great near the earth's surface that this case is not of much practical consequence, and indeed the effect of inhomogeneities in the ground is generally likely to be much less marked than for vertical polarization.

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